Credulity, lies, and costly talk*

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April 2006

Abstract

This paper studies a model of strategic communication by an informed and upwardly biased sender to one or more receivers. Applications include situations in which (i) it is costly for the sender to misrepresent information, due to legal, technological, or moral constraints, or (ii) receivers may be credulous and blindly believe the sender's recommendation. In contrast to the predictions obtained in Crawford and Sobel's [9] benchmark cheap talk model, our model admits a fully separating equilibrium, provided that the state space is unbounded above. The language used in equilibrium is inflated and naive receivers are deceived.

JEL classification: C72; D82; D83

Keywords: Signaling; Cheap talk; Misreporting costs; Lying; Deception; Naive receivers; Inflated communication; Full revelation

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*This article supersedes Ottaviani and Squintani's 2002 paper "Non-fully strategic information transmission" and Kartik's 2005 paper "Separating equilibria with costly misreporting".

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"The credulity of dupes is as inexhaustible as the invention of knaves." — *Edmund Burke*

1 Introduction

Conflicts of interest plague communication in many situations, often resulting in language inflation and deception. For example, there is empirical evidence that financial analysts provide overoptimistic recommendations that deceive some investors.[1] In addition, there appear to be systematic differences across investors in their responses to the same recommendation.[2] These findings suggest that some investors might be credulous and naively believe what they hear.

Building on Crawford and Sobel's [9] (hereafter, CS) "cheap talk" model, this paper formulates a model that encompasses situations in which investors naively believe in the analyst's recommendation. We show that the credulity of the investor turns cheap talk into "costly talk", naturally resulting in language inflation and deception. In addition to financial advice, our model can be applied to various other situations, such as advertising and academic evaluation. Advertisers typically aim at persuading their audience to purchase more by overemphasizing the qualities of the products they promote. Similarly, academic advisers tend to assign inflated grades and write overly positive recommendation letters to help their students get jobs that are better than truly deserved (Rosovsky and Hartley [24]).

It is difficult to reconcile language inflation and deception with the theoretical predictions of the baseline equilibrium model of cheap talk.[3] In CS's model, a privately informed *sender* (e.g., the analyst) sends a *message* (e.g., a recommendation) to a *receiver* (e.g., the investor) with misaligned preferences. The receiver then takes an *action* (e.g., purchase of a stock), which could depend on the particular message sent. A key feature of cheap talk is that the message affects the players' payoffs only indirectly, through the inference made by the receiver about the sender's private information. Being rational and aware of the conflict of interests, the receiver takes an equilibrium action that is unbiased, conditional on the information transmitted. In equilibrium, the sender's bias and the

[1] See Michaely and Womack [20], who attribute the overoptimism in the recommendations to conflicts of interest between analysts and investors. The finance literature has identified at least three sources of these conflicts of interest. First, conflicts are possibly created by the analysts' incentives to generate investment-banking business (Michaely and Womack [20]). Second, analysts might want to increase the brokerage commissions for the trading arms of their financial firms (Jackson [14]). Third, analysts might tend to release favorable reports that please management in order to gain access to internal information from the firms they cover (Lim [16]).

[2] As argued by Malmendier and Shanthikumar [18], these differences seem to indicate heterogeneity in the level of strategic sophistication among investors. In controlled laboratory experiments, Dickhaut et al. [10] and Cai and Wang [5] document heterogeneous strategic sophistication in the responses to the same message.

[3] In this context, we view deception as the act of inducing false beliefs by means of communication, and exploiting them to one's own advantage. Such false beliefs are clearly incompatible with traditional equilibrium analysis. Deception in this interpretation is distinct from the notion that a player may choose not to disclose private information in order to exploit the imprecise—but not incorrect—belief induced in a counterpart (e.g., bluffing in poker).

associated incentive to deceive the receiver are, in fact, self-defeating. In the context of financial advice, this implies that, on average, no investor should be deceived. Moreover, the language used in equilibrium is arbitrary, since cheap talk messages are fully interchangeable.

In contrast, the costly signaling model of strategic communication developed in this paper naturally leads to language inflation. In our costly talk model, the message sent affects the sender's payoff, not only indirectly through a rational receiver's inference, as in a cheap talk model, but also directly. This direct dependence is our key departure from CS's cheap talk model and can stem from various sources. For applications, we focus on two interpretations. First, following Kartik [15], the sender may suffer a disutility from misreporting or lying about her private information. This could be due to fabrication costs, legal penalties, or moral constraints. Plainly, the presence of such costs transforms the game from one of cheap talk to costly signaling.

Second, the direct dependence of the sender's utility on messages can stem from the receiver(s) being, at least partially, non-strategic. Instead of acting on the basis of equilibrium beliefs, such receivers are somewhat credulous: either they naively believe that the sender's message represents the truth and take their preferred action accordingly, or, more generally, they use some non-equilibrium-based rule to map messages into actions.[4] We say that a receiver is credulous or naive because she is unable to anticipate the equilibrium interpretation of the communication language. Instead, her interpretation of the language is exogenous, perhaps based on some general social principles, such as truth-telling, or some boundedly rational computation that results in a cautious interpretation of the sender's report. The sender, being sophisticated, understands the mapping from messages to actions that are used by these naive receivers, and exploits their credulity in order to deceive them.[5] The mechanical response to messages by such receivers induces a direct dependence of the sender's payoff on his recommendation, thereby resulting in a game of costly signaling.

At a technical level, we also depart from CS by making different assumptions about the domain of a sender's private information. CS assume that the set of sender types (i.e., the *state space*) is bounded both above and below. Adopting the common convention that the sender is biased upwards, this paper instead considers settings in which the state space is unbounded above (and either bounded

[4] In studying lying costs or naive receivers, it is implicit that messages have an exogenous natural meaning. We suppose that the message space is identical to the space of the sender's private information, so that a message can be interpreted as a literal statement about the sender's type. The concept of natural meaning has been introduced in a different context by Farrell [12].

[5] We view the asymmetry of the sender being sophisticated while some receivers being naive or credulous as appropriate for many applications, such as the ones discussed earlier that involve financial analysts and advertisers communicating to a market.

or unbounded below). We believe that this is a sensible assumption in a number of applications, for example when modeling a financial analyst's prediction of a stock price or an adviser's evaluation of student ability.

Our main contribution in this setting is to show that, under broad conditions, there exist separating equilibria which feature inflated communication.[6] In a separating equilibrium, the sender's message completely reveals her private information. In the equilibria we identify, the message sent by the sender has a literal meaning that is inflated, i.e., a literal meaning higher than the true state of the world. Nevertheless, a sophisticated receiver correctly infers the true state by inverting the observed message according to the equilibrium language. A credulous receiver, instead, interprets the equilibrium messages with some non-equilibrium-based rule and is accordingly deceived, taking biased actions. These findings provide a striking contrast to CS's key results that, in a fully-strategic and pure cheap talk setting with a conflict of interests, every equilibrium is partitional, some information must be lost, and the receiver's actions are unbiased in expectation.[7]

The intuition behind our results can be illustrated as follows. Suppose first that the sender has lying costs that are increasing and convex in the magnitude of the lie. An inflated communication separating equilibrium exists, because even though the sender can induce the (sophisticated) receiver to take a more favorable action by deviating to a higher message, this gain can be exactly offset by the marginal increase in the lying cost when the equilibrium message is already sufficiently inflated. On the other hand, suppose instead that there are no lying costs but that some receivers are wholly credulous, simply believing the literal message they receive. In the separating equilibrium, for any given state the message is so inflated that the credulous receivers are deceived to take an action that is even higher than the one that is ideal for the sender. Sophisticated receivers decode the message and take their most-preferred action, which is lower than the sender's ideal action. Thus, by deviating to a higher (lower) message, the sender gains through the induced response from sophisticated (credulous) receivers, but is hurt by the undesirable response from credulous (sophisticated) receivers. By exactly offsetting these gains and losses, a separating equilibrium is sustained.

Our formal results are derived in a quite general setting with broad and abstract conditions that are best discussed only after introducing the model in Section 2. In the remainder of this introduction, we describe each of our three results on the existence of separating equilibria, focusing on particular

[6] As explained in Sections 3 and 5, our results hinge critically on the assumption that the state space is unbounded *above*. When the state space is bounded above, there is an equilibrium that is fully revealing only up to a threshold smaller than the upper bound of the state space (see Kartik [15] or Ottaviani and Squintani [23]).

[7] Although the state space is bounded in the original CS model, the aforementioned results extend straightforwardly to an unbounded state space.

applications that are covered by the assumptions in each case.

Theorem 1 applies to settings in which the sender is interested in the *average* response of a population of receivers characterized by heterogeneous strategic sophistication. As previously noted, a prime application is a financial analyst communicating to a market of investors. We demonstrate that in such cases there is a unique non-decreasing, differentiable separating equilibrium. This equilibrium has the following important property: in every state of the world, the sender induces a belief in the naive receivers such that the average population response is in fact his bliss point. That is, the sender can achieve his first-best outcome in such a setting, even though the sophisticated receivers correctly infer the state of the world in equilibrium and there is a conflict of interest.

Theorems 2 and 3 apply to communication games where the sender communicates to a single receiver, but either bears a cost from misrepresenting her type, or the receiver is possibly naive and credulous. For technical reasons explained in Section 3, we need to further distinguish between cases where the state space is bounded below and cases where it is unbounded below. These two cases are relevant for different applications. For example, when thinking about a real estate agent making forecasts about the future price of a property to a potential buyer, there is a natural lower bound of zero on the state space (but no natural upper bound). In contrast, if an expert is reporting to a decision-maker on the one-dimensional ideological position of a terrorist group, there is no natural lower or upper bound. When the state space is bounded below, Theorem 2 shows that there is a unique differentiable separating equilibrium that satisfies the standard *Riley condition*.[8] On the other hand, when the state space is unbounded below, there is no such "initial condition", and some restrictions are needed to guarantee the existence of separating equilibria. Theorem 3 derives existence for a class of widely-used sender preferences, viz. those that can be represented as loss functions which depend on the distance between the receiver's action and the sender's state-dependent bliss point.

In addition to their theoretical interest, our results have positive and normative implications. For example, it is unclear why a mandated policy of so-called "Chinese walls" would be necessary in the benchmark model of information transmission between a financial corporation and an investor. Since in CS's setting the bias only reduces the amount of information that can be credibly transmitted, without systematically biasing final outcomes, it would be in the firm's own interest to minimize the conflict of interest by separating its research and investment-banking divisions into different corporations. Instead, our model suggests a possible rationale for policy intervention. In the presence of naive receivers, the firm is able to systematically deceive a fraction of the investors, and may not

[8]The Riley condition requires that the lowest type of sender not engage in any (Pareto) inefficient signaling.

spontaneously adopt this policy. Mandated Chinese walls would then protect naive investors, without affecting sophisticated investors who are able to decode the firm's communication in any case.

In the literature, Crawford [8] is the first paper to analyze the impact of bounded rationality on communication outcomes. While Crawford [8] focuses on the representation of intentions (about future actions) in an asymmetric matching-pennies game, we consider the transmission of private information. Both Crawford [8] and our paper obtain deception somewhat directly from the introduction of non-equilibrium players, but are interested in the implications for the behavior of fully rational players.[9] We defer to Section 4 the discussion of other related literature, in particular the connection with some communication models that allow for additional layers of uncertainty.

2 Model and applications

We develop a general model of communication between an informed agent and one or more uninformed agents. After introducing our setting, we illustrate how it subsumes several natural variations of an unbounded state space version of CS's classic model of cheap talk.

A sender (S) is privately informed about a state of the world, $x \in X \subseteq \mathbb{R}$, distributed according to the full support cumulative distribution function F. We sometimes refer to x as the sender's type. After observing the state, the sender chooses a message, $m \in M$, to communicate to one or more receivers (R). We assume that the message and state space are the same, so that the message sent can be thought of as a literal statement or a submitted report about the state of the world. The state (and message) space is unbounded above, and may be either bounded or unbounded below. We denote by $\langle a, \infty)$ the open interval (a, ∞) if $a = -\infty$, or the semi-closed interval $[a, \infty)$ if $a > -\infty$. The state and message space are $X = M = \langle \underline{x}, \infty)$, where $\underline{x} \geq -\infty$.

A (pure) strategy for the sender is a mapping $\mu : X \to M$. Our interest is in equilibria where the sender plays a one-to-one strategy, thereby completely revealing her private information through the message sent. Such a strategy for the sender is a *separating* strategy, sometimes referred to in the literature as "fully separating" or "fully revealing". To describe such equilibria, it suffices to focus on a reduced form payoff function for the sender, $U(x, \hat{x}, m)$, rather than making explicit assumptions about the preferences or play of the receivers. This function denotes the sender's payoff when the state of the world is $x \in X$, he sends message $m \in M$, and the (strategic) receivers believe the state

[9] Ettinger and Jehiel [11] develop a theory of deception, based on the idea that some players may only be able to understand strategies "coarsely", although the interpretation is always, at least partially, equilibrium based.

of the world to be $\hat{x} \in X$. We maintain throughout our analysis the following regularity conditions on U (as usual, subscripts on functions denote derivatives).

Condition A. *The function U satisfies:*

(A.1) $U(x, \hat{x}, m)$ *is* $\mathcal{C}^2$ *on* $X \times X \times M$.[10]

(A.2) *For all* x, $U_3(x, x, m) = 0$ *has a unique solution in* m, *denoted* $m^*(x)$, *which maximizes* $U(x, x, m)$, *and* $U_{33}(x, x, m^*(x)) < 0$.

(A.3) $\lim_{x \to +\infty} m^*(x) = \infty$. *When* $\underline{x} = -\infty$, $\lim_{x \to -\infty} m^*(x) = -\infty$. *When* $\underline{x} > -\infty$, $U(x, \underline{x}, m) \leq U(x, \underline{x}, m^*(\underline{x}))$ *for all* x *and* $m < m^*(\underline{x})$.

(A.4) *For all* $(x, \hat{x}, m)$, $U_{12}(x, \hat{x}, m) > 0$.

(A.5) *For all* $(x, \hat{x}, m)$, $U_{13}(x, \hat{x}, m) > 0$.

(A.1) is a smoothness assumption. (A.2) says that for each state x, there is a unique utility-maximizing message $m^*(x)$ when the inference is correctly made that the state is x. That is, $m^*(x)$ is the message the sender would choose under complete information. Since U is twice continuously differentiable, m^* is a continuously differentiable function, and it follows that $U_3(x, x, m) < 0$ for $m > m^*(x)$, whereas $U_3(x, x, m) > 0$ when $m < m^*(x)$. We note that (A.2) is weaker than concavity of $U(x, x, m)$ in m for all x. It is important to emphasize that (A.2) implies that messages are payoff relevant, and not cheap talk. In a pure cheap talk model, $U_3 = 0$.

Condition (A.3) allows us to deal with the possibility of messages off the equilibrium path. When the state space is unbounded below, the assumption that m^* is unbounded above and below allows us to focus on increasing equilibrium strategies μ that are onto (and hence use all available messages). When the state space has the lower bound $\underline{x} > -\infty$, we will study increasing strategies μ such that $\mu(\underline{x}) = m^*(\underline{x})$. Because m^* is unbounded above as $x \to \infty$, all messages above $m^*(\underline{x})$ are used in equilibrium. The payoff condition in (A.3) ensures that if no type of sender has an incentive to mimic the lowest type, then there will not be an incentive to send a message lower than $m^*(\underline{x})$, if such messages are met with the inference of coming from the lowest type. Note that given (A.2), this part of (A.3) is automatically satisfied if $m^*(x)$ is strictly increasing, and, moreover, $U(x, \hat{x}, m)$ is separable in m and $\hat{x}$.

[10] That is, there exists an open set containing $T \times T \times M$ and a twice continuously differentiable function on this open set that equals U on $T \times T \times M$.

Conditions (A.4) and (A.5) are Spence-Mirrlees single-crossing conditions. (A.4) says that the sender's marginal gain from inducing a higher belief about the state is higher when the true state is higher. (A.5) requires that the sender's marginal gain from sending a higher message is higher when the true state is higher.

In addition to Condition (A), which we always maintain, two mutually exclusive sets of conditions—(B) and (C)—will allow us to find separating equilibria.

Condition B. *The function U satisfies:*

(B.1) *For all (x, m), sign $[U_2(x, x, m)] = $ sign $[U_3(x, x, m)]$.*

(B.2) *For all x, ${m^*}'(x) > 0$.*

(B.1) requires that when the state is correctly inferred as x, inflating the message m or inducing a belief that the state is higher than x affects the sender's utility in the same direction. Note that in the presence of (A.2), this implies that $U_2(x, x, \cdot)$ is single peaked around $m^*(x)$. It is worth contrasting this with most signaling and cheap talk models, where the assumptions imply that the sender would always like to induce a belief that the state is higher than the true state, regardless of the message used (see condition (C.1), below). (B.2) says that the optimal message $m^*(x)$ is strictly increasing in the state x. This will be used to ensure that the communication strategy of $\mu = m^*$ is a separating and increasing strategy.

Condition C. *The function U satisfies:*

(C.1) *For all (x, m), $U_2(x, x, m) > 0$.*

(C.2) *For all x, there are $c_2, c_3 > 0$ such that for all m: (i) $U_2(x, x, m) < c_2$ and (ii) $U_{33}(x, x, m) \geq 0 \implies |U_3(x, x, m)| > c_3$.*

(C.1) is a generalization of a standard assumption of traditional Spencian signaling models. There, the sender's payoff is assumed to be strictly increasing in her perceived type, i.e. $U_2(x, \hat{x}, m) > 0$ for all $x, \hat{x}, m$. Here, we only require it locally around $x = \hat{x}$. As already noted, (C.1) and (B.1) are mutually exclusive under (A.2). (C.2) is a technical condition on the boundedness of derivatives.

Conditions (A) and (C) bear many similarities with Mailath [17], who studies a bounded type space. In particular, he also employs (A.1), (A.2), (A.5), and (C.2).[11] On the other hand, the set of payoff conditions (B) that we have introduced is novel.

2.1 Applications of costly talk

Consider an unbounded state space version of the fully-strategic, pure cheap talk model of CS. Messages are payoff irrelevant to the sender, i.e. $U_3(x, \hat{x}, m) = 0$, for any $x, \hat{x}, m$. (Hence, our condition (A.2) is not satisfied.) The state space is either $X = (-\infty, \infty)$ or $X = [\underline{x}, \infty)$. There is one receiver, who takes an action $y \in \mathbb{R}$ after receiving the sender's message. The receiver's utility is denoted by $U^R(y, x)$, and the sender's utility is expressed as $U^S(y, x, b)$, with $b \in \mathbb{R}$ being a bias parameter that is common knowledge among the players. Both U^R and U^S are twice continuously differentiable. For each $i = S, R$, player i's utility satisfies $U^i_{11} < 0$ as well as the single-crossing condition $U^i_{12} > 0$. Moreover, there exist bliss-point functions $y^R(x)$ and $y^S(x, b)$ such that $U^R_1(y^R(x), x) = 0$ and $U^S_1(y^S(x, b), x, b) = 0$. Note that the assumptions on U^R and U^S imply that $y^S_1(x, b) > 0$ and $y^{R\prime}(x) > 0$.

Translated into our formulation, CS sender's payoff can be written as $U(x, \hat{x}, m) \equiv U^S\left(y^R(\hat{x}), x, b\right)$, since messages are payoff irrelevant, and the receiver takes action $y^R(\hat{x})$ when she believes the state to be $\hat{x}$. When the state space is bounded, a key result of CS is that if the players' bliss points never coincide, i.e. $y^S(x, b) \neq y^R(x)$ for all x, then any equilibrium outcome mapping from states to receiver actions is a step function, and hence every equilibrium is partitional. Essentially, this fundamental result extends to the unbounded state space case, where every equilibrium is partitional if there is a uniform bound $\psi > 0$ such that $\left|y^S(x, b) - y^R(x)\right| \geq \psi$ for all x.[12] This condition requires that the sender be biased either upwards or downwards for all states of the world. We shall henceforth assume that he is biased upwards, i.e., $y^S(x, b) - y^R(x) \geq \eta > 0$ for all x, and assume that $U^S_{13}(y, x, b) > 0$.

We now argue that natural modifications of CS's assumptions transform their cheap talk game, in which $U_3(x, \hat{x}, m) = 0$ for all $x, \hat{x}, m$, into a costly signaling (or "costly talk") game, in which our condition (A.2) applies instead. In particular, we introduce three variations of the unbounded state space CS model that satisfy our payoff conditions (A) as well as either (B) or (C).

[11] Our condition (C.1) is weaker than his "belief monotonicity" condition. (A.3) has no analog in his analysis; (A.4) substitutes for the role of his "single-crossing" condition in $\hat{x}$ and m.

[12] The proof of this result is omitted as it is an immediate extension of Lemma 1 in CS. The notion of equilibrium referred to here is Bayesian Nash or any stronger concept. Note that the uniform bound condition is trivially satisfied when the state space is bounded.

2.1.1 Application of conditions (A) and (B)

Application 1. (Communication to a pool of receivers, some of whom are naive) In several economic applications of the CS game, the sender makes a recommendation to a pool of receivers and cares about the *average* aggregate response of the receivers. For example, in the context of stock recommendations, the analyst often announces her forecast to the entire market and cares about the market's aggregate response following the announcement. This scenario can be equivalently represented as a CS communication game with only one receiver if all the receivers are fully strategic and their strategy depends only on the message received. But, as we will later show, these two models are substantially different when the market is composed of agents with heterogeneous sophistication.

To account for heterogeneous sophistication, suppose that a fraction α of receivers is either partially or fully naive, whereas a fraction $1-\alpha$ is strategic. Fully naive receivers blindly believe the sender's report. Upon hearing message m, they believe that $x = m$. Partially naive receivers are aware of the possibility of being cheated, but are still unable to formulate equilibrium beliefs. Upon receiving any message, m, they formulate a dis-equilibrium estimate $\chi(m)$ of the true state of the world such that $\chi(m) \in \langle \underline{x}, m]$. We assume that χ is strictly increasing, continuously differentiable, unbounded above, and also unbounded below if $\underline{x} = -\infty$.[13] If strategic receivers believe that the state is $\hat{x}$, the utility of the sender of type x who sends message m is:

$$U(x, \hat{x}, m) \equiv U^S\left((1-\alpha)\, y^R(\hat{x}) + \alpha y^R(\chi(m)), x, b\right). \tag{1}$$

Fixing $\alpha > 0$ and $b > 0$, we develop the application under the restriction on the players' utilities that the functions $y^R(x)$ and $y^S(x,b) - (1-\alpha)\, y^R(x)$ are strictly increasing in x and unbounded. For example, this is the case when $y_1^S(x,b) \geq y^{R\prime}(x) > \eta$ for some uniform bound $\eta > 0$: the sender's bliss point is at least as sensitive to a change in the state as the receivers' bliss points.[14] As a technical condition on derivatives, we also assume that $y^{R\prime}$ and χ' are bounded away from ∞. (This would be automatically satisfied were the state space bounded.)

The following Lemma shows that the presence of naive agents in the pool of receivers turns the cheap talk CS game into a game of costly signaling that satisfies our conditions (A) and (B).

[13] Monotonicity of χ is natural: it says that a naive receiver believes the state is higher when the message she receives is higher. If χ were to be bounded, say above, there would be a state $\bar{x}$ such that a naive receiver would *never* believe that the state x is larger than $\bar{x}$, regardless of the message received. This is equivalent to working with a bounded type space (see Kartik [15] or Ottaviani and Squintani [23]).

[14] Note that this holds for any class of CS preferences where $y^R(x) = x$ and $y^S(x) = x + b$, such as the commonly used "quadratic loss" formulation where $U^R(y,x) = -(y-x)^2$ and $U^S(y,x,b) = -(y-x-b)^2$.

Lemma 1. *The game between a sender who can lie costlessly and a pool of partly naive receivers satisfies conditions (A.1)–(A.5) and (B.1)–(B.2) under the stated assumptions. Moreover, for all* x*,* $m^*(x) > x$*.*

That $m^*(x) > x$ is intuitive: if the strategic receivers know the true state and play $y^R(x)$, then the sender inflates her message to make the naive receivers' response larger than their bliss points $y^R(x)$, so that the average receivers' response equals $y^S(x,b) > y^R(x)$.

2.1.2 Applications of conditions (A) and (C)

We will discuss two simple and intuitive modifications of the CS game that are subsumed under the following separable utility specification:

$$U(x,\hat{x},m) \equiv (1-\alpha)\,U^S\left(y^R(\hat{x}),x,b\right) + (\alpha+k)G\left(g(m),x,b\right), \tag{2}$$

where $\alpha, k \geq 0$ with at least one strictly positive. U^S is the CS utility function for the sender, and G and g are functions that will capture the modification to the game that transforms it from one of cheap talk to costly signaling.

Some reasonable assumptions are needed to ensure that the above specification (2) satisfies conditions (A) and (C). First, $\sup_x U_2^S(y^R(x),x,b)\, {y^R}'(x) < \infty$. The function $g : \mathbb{R} \to \mathbb{R}$ is assumed to be $\mathcal{C}^2$, strictly increasing and onto, with derivative bounded away from 0 and ∞: there exist constants γ_1 and γ_2 such that $0 < \gamma_1 < g' < \gamma_2 < \infty$. The function G has the same properties as U^S in the CS game: G is $\mathcal{C}^2$, $G_{12} > 0$, $G_{13} \geq 0$, and $G_{11} < 0$. For each x, there is $y^*(x,b)$ such that $G_1(y^*(x,b),x,b) = 0$, and there is γ_3 such that $0 < \gamma_3 < y_1^*$.

Lemma 2. *Under the stated assumptions, the representation* (2) *satisfies conditions (A.1)–(A.5) and (C.1)–(C.2).*

The use of Lemma 2 is illustrated in the following applications. We wish to emphasize that since conditions (B.1) and (C.1) are mutually exclusive under condition (A), the ensuing applications are substantially different from Application 1.

Application 2. (Communication to a single receiver who may be naive) Consider a standard CS game, but assume that the receiver is naive with probability $\alpha \in (0,1)$ and is strategic with probability $1-\alpha$.[15] As in Application 1, a naive receiver formulates an estimate $\chi(m)$ of the

[15] In fact, it is not necessary that the receiver actually be truly naive with positive probability. It is enough that the sender believes (perhaps wrongly) that the receiver could be naive.

true state such that $\chi(m) \in \langle \underline{x}, m]$, and we maintain our previous assumptions on χ. If the true state is x, the strategic receiver believes that the state is $\hat{x}$, and message m is sent, the sender's payoff is

$$U(x, \hat{x}, m) \equiv (1-\alpha)\, U^S\left(y^R(\hat{x}), x, b\right) + \alpha U^S\left(y^R(\chi(m)), x, b\right).$$

We assume the technical conditions that ${y^R}'$ and χ' are each bounded away from 0 and ∞, and that $U_2^S(y^R(x), x, b)\, {y^R}'(x)$ is bounded away from ∞. (All these conditions would automatically hold when the state space is bounded.)

It is evident that this application satisfies utility specification (2) and the assumptions for Lemma 2 by setting $k=0$, $G(y,x,b) = U^S(y,x,b)$, and $g(m) = y^R(\chi(m))$. Note that in this application, $m^*(x) > x$. If the receiver knows the state of the world when strategic, then she plays $y^R(x)$ regardless of the sender's message. The sender would then optimally send an inflated message to induce a naive receiver to take the sender's preferred action $y^S(x, b)$.

Application 3. (Communication by a sender with misreporting costs) Suppose that the sender communicates to one or many receivers who are fully strategic with probability one, but the sender suffers a cost of lying or misreporting. For example, the sender may have a preference for honesty, or may face technological or verifiability constraints in forging a mendacious report. Following Kartik [15], we let the lying costs be $kC(m,x)$ where $C(\cdot, x)$ is a loss function corresponding to the state x, and $k > 0$ parameterizes the intensity of the lying cost. Assume that $C_{11} > 0 > C_{12}$, and that for any x, $C_1(x,x) = 0$. Hence, lying costs are convex around the truth. In this setting, we can write the sender's utility as

$$U(x, \hat{x}, m) \equiv U^S(y^R(\hat{x}), x, b) - kC(m, x).$$

As before, we assume the technical condition that $\sup_x U_2^S(y^R(x), x, b)\, {y^R}'(x) < \infty$. It is clear that this application satisfies utility specification (2) and the assumptions for Lemma 2 by setting $\alpha = 0$, $G(m,x,b) = -C(m,x)$, and $g(m) = m$. Note that here, $m^*(x) = x$: if the receiver knows the state, then the sender chooses to report the truth, as there is no possible benefit from lying.

3 Separating equilibria

The solution concept to apply in signaling games is perfect Bayesian equilibrium (Fudenberg and Tirole [13]). This requires the usual conditions of belief consistency via Bayes rule, and incentive compatibility of strategies given beliefs and opponents' play. Our interest concerns separating perfect Bayesian equilibria, which are equilibria where the sender's strategy μ is one-to-one and hence fully reveals her type. Denote the receivers' beliefs by $\beta : M \to \Delta(X)$, where $\Delta(X)$ is the set of probability distributions on X.[16] Letting $\mu(X) \equiv \bigcup_{x \in X} \mu(x)$, Bayes rule requires that for all $m \in \mu(X)$, $\beta(m) = \mu^{-1}(m)$. For any $m \in X \setminus \mu(X)$, any beliefs are permissible, but we will demonstrate that we can restrict attention to point mass beliefs ($\beta : M \to X$) without loss of generality for the purpose of our results. These considerations lead to the following definition.

Definition 1. *A separating equilibrium is one-to-one strategy, $\mu : X \to M$, and beliefs, $\beta : M \to X$, such that:*

1. (Belief consistency) *for any $m \in \mu(X)$, $\beta(m) = \mu^{-1}(m)$.*

2. (Incentive compatibility) *for any $x \in X$, $\mu(x) \in \arg\max_{m \in M} U(x, \beta(m), m)$.*

It should be noted that our results will concern differentiable and strictly increasing separating equilibria.[17] Differentiability is natural. The reason we look for increasing equilibria is that in applications, including those outlined in Section 2, it is typically the case that higher sender types prefer higher messages, and, moreover, the sender would like to be perceived as a higher type than he truly is. In such cases, decreasing equilibrium strategies seem relatively implausible.

We should be clear about what it means for language to be inflated. An equilibrium with sender's strategy μ displays *inflated communication* if $\mu(x) > x$ for all x (with the possible exception of the finite lower bound $\underline{x}$ if it exists). This definition is natural in our context where messages represent literal statements or reports about the state of the world.

3.1 Communication to a pool of receivers

Recall that condition (B) was shown to apply when the sender communicates to a pool of receivers with heterogeneous strategic sophistication. Our first main result is that in this case, there exists a unique non-decreasing differentiable separating equilibrium. In any differentiable, separating equilibrium, the necessary first-order condition for optimality is

$$U_2(x, \mu^{-1}(m), m) \left(\mu^{-1}(m)\right)' + U_3(x, \mu^{-1}(m), m) = 0. \tag{3}$$

[16] Throughout, when referring to receivers' beliefs and actions, we implicitly restrict attention to the strategic receivers. Also, as is standard, if there are multiple strategic receivers, we require them to hold the same beliefs upon observing any message.

[17] Such properties refer to properties of the sender's strategy in equilibrium. Similarly, all claims to uniqueness of equilibria within some class refer to uniqueness of sender strategies (and not out-of-equilibrium beliefs, in particular).

Substituting $\mu^{-1}(m) = x$ and $\left(\mu^{-1}(m)\right)' = 1/\mu'(x)$, and rearranging, we obtain

$$U_3(x, x, \mu(x))\mu'(x) + U_2(x, x, \mu(x)) = 0. \tag{FOC}$$

In light of assumptions (A.2) and (B.1), there is a unique solution to this equation such that $\mu'(x) > 0$: the function μ such that $U_2(x, x, \mu(x)) = U_3(x, x, \mu(x)) = 0$. By definition, such a sender's strategy μ coincides with m^*. Verifying incentive compatibility of the solution obtained with the first-order approach leads us to the following result.

Theorem 1. *Under conditions (A) and (B), there is a unique non-decreasing differentiable separating equilibrium. It is given by $\mu(x) = m^*(x)$, and is therefore strictly increasing.*

To see the intuition behind the result, consider the case where the sender can lie costlessly, but faces a pool of receivers some of whom are fully naive (Application 1). Suppose there is an equilibrium where μ is invertible, non-decreasing, and, for simplicity, onto. In equilibrium, the strategic receivers correctly invert the message, determine the state $x = \mu^{-1}(\mu(x))$, and play $y^R(x)$, whereas the naive receivers believe the message $\mu(x)$ and play $y^R(\mu(x))$. The key point is that the sender only cares about the average receiver response $(1-\alpha)\, y^R(x) + \alpha y^R(\mu(x))$. For any $\alpha > 0$, because the message space M coincides with the unbounded-above state space X, it is possible to find $\mu(x)$ large enough so that the average receiver response exactly equals the sender's bliss point $y^S(x, b)$, and hence the sender has no incentive to deviate from equilibrium.[18] Such $\mu(x)$ must be $m^*(x)$ by the definition of m^*. This is indeed invertible—specifically, strictly increasing—by condition (B.2). Hence, communication is inflated, as $m^*(x) > x$ for all x, by Lemma 1. Perhaps surprisingly, the sender achieves her bliss point in equilibrium for any state of the world, x, regardless of the fraction of naive receivers, α. This is a consequence of two properties: first, the sender only cares about the average response of the receivers; second, due to the unboundedness of the state space, the sender can deceive naive receivers as much as necessary to attain her bliss point.

To substantiate Theorem 1 and the above discussion, we explicitly compute a simple example with the widely used quadratic utility formulation.

Example 1. Suppose that naive receivers are fully naive, i.e., $\chi(m) = m$ for every m, and that the receivers' and sender's utilities are quadratic loss functions with bliss points x and $x + b$, respectively.

[18] In the general setup, this is guaranteed by (A.2) and (B.1) for deviation to messages used on the equilibrium path. When $X = (-\infty, +\infty)$, all messages are on path by (A.3). When $X = [\underline{x}, +\infty)$, with $\underline{x} > -\infty$, we assign beliefs $\beta(m) = \underline{x}$ to all off-path messages $m \in [\underline{x}, m^*(\underline{x})]$, if there are any. (A.3) assures that there is no sender type who strictly benefits by playing such a message over his equilibrium message.

That is, $U^R(y,x) = -(y-x)^2$ and $U^S(y,x,b) = -(y-(x+b))^2$. Suppose that in equilibrium, the sender adopts a differentiable invertible function μ as her communication strategy. When a message m is sent, a strategic receiver believes that the state is $\mu^{-1}(m)$ and a naive receiver plays the action $y^R(m) = m$. Hence the sender will not deviate from the strategy μ only if for any x, $\mu(x) \in \arg\max_m -\left((1-\alpha)\mu^{-1}(m) + \alpha m - (x+b)\right)^2$. The first-order condition is

$$-2\left((1-\alpha)\mu^{-1}(m) + \alpha m - (x+b)\right)\left((1-\alpha)\left(\mu^{-1}(m)\right)' + \alpha\right) = 0.$$

By substituting $\mu^{-1}(m)$ with x and m with $\mu(x)$ we obtain the differential equation

$$-2(\alpha(\mu - x) - b)((1-\alpha)\frac{1}{\mu'} + \alpha) = 0.$$

It is easy to see that the unique non-decreasing solution μ is

$$\mu(x) = m^*(x) = x + \frac{b}{\alpha},$$

and this is the strategy in the unique differentiable, non-decreasing, separating equilibrium.[19]

This strategy may be interpreted as revealing the actual state of the world and inflating the communication by an amount b/α. The factor by which communication is inflated is inversely proportional to the fraction of naive receivers in the population. It immediately follows that $\mu(x)$ increases in b and decreases in α. When α shrinks to 0, $\mu(x)$ diverges to infinity pointwise, and the naive receivers' utility diverges to minus infinity. Because the average receiver response is $\alpha(x + b/\alpha) + (1-\alpha)x = x + b$, the sender achieves her bliss point for all x independent of the parameter α. □

3.2 Communication to a single receiver: lower bounded state space

We now turn to showing the existence of a separating equilibrium under the set of conditions (C), assuming that there is a lower bound on the state space, $\underline{x} > -\infty$. As in the previous sub-section, any differentiable, separating equilibrium sender's strategy μ must satisfy the necessary first-order condition for optimality (FOC). But unlike when (B.1) holds, under (C.1) there is no such m such that $U_2(x,x,m) = 0$. Hence (FOC) cannot be satisfied with $U_3(x,x,\mu(x)) = 0$. We can thus rearrange the expression and obtain the following differential equation in μ

$$\mu'(x) = -\frac{U_2(x,x,\mu(x))}{U_3(x,x,\mu(x))}, \quad \text{(DE)}$$

where $U_3(x,x,\mu(x)) = 0$ describes the singularity in the field.

[19] When the state space is unbounded below, all messages are on the equilibrium path; when it has a lower bound $\underline{x} > -\infty$, we complete the equilibrium by assigning the off-path beliefs $\beta(m) = \underline{x}$ for all $m < \mu(\underline{x}) = m^*(\underline{x})$.

Theorem 2. *Suppose that $\underline{x} > -\infty$. Under conditions (A) and (C) and the initial value condition $\mu(\underline{x}) = m^*(\underline{x})$, there is a unique differentiable separating equilibrium. In this equilibrium, μ solves (DE), is strictly increasing, and $\mu(x) > m^*(x)$ for all $x > \underline{x}$.*

Remark 1. Throughout, when the type space is bounded below, we focus on the initial condition of $\mu(\underline{x}) = m^*(\underline{x})$. There may be separating (and differentiable) equilibrium strategies where $\mu(\underline{x}) \neq m^*(\underline{x})$,[20] but in signaling games with multiple separating equilibria, it is common to focus on the one where the lowest type sends the same signal as it would under complete information. This is often referred to as the *Riley outcome*. In the current setting, this requires that $\mu(\underline{x}) = m^*(\underline{x})$.

The main part of the proof of Theorem 2 extends the analysis of Mailath [17] to an unbounded state space.[21] By studying the inverse differential equation of (DE), we first show that there exists a unique increasing local solution to (DE) under the initial condition $\mu(\underline{x}) = m^*(\underline{x})$.[22] We then prove that under condition (C.2), this local solution can be uniquely extended to the whole domain $X = [\underline{x}, \infty)$. This solution lies strictly above $m^*(x)$ for all x sufficiently close to $\underline{x}$, since (DE) requires that $\mu'(x) \to \infty$ as $x \to \underline{x}$, by (A.2), (C.1), and (C.2). But then, it must be that $\mu(x) > m^*(x)$ for all $x > \underline{x}$; otherwise, by continuity, μ must cross m^* at some point $\hat{x}$, which it cannot do because (DE) requires that $\mu'(x) \to \infty$ as $x \to \hat{x}$. By inspection of (DE), the strategy μ is strictly increasing, because of (C.1), (A.2), and $\mu(x) > m^*(x)$ for all $x > \underline{x}$. Since the only off-the-equilibrium-path messages (if any) are those below $m^*(\underline{x})$, the treatment of off-the-equilibrium-path beliefs are analogous to Theorem 1: we assign the belief that the sender is of the lowest type when observing any such message. Incentive compatibility of this equilibrium is verified using conditions (A.3), (A.4), and (A.5).

Theorem 2 yields a corollary that there is a fully-revealing equilibrium in the modifications of the CS game where lying entails a cost of sensitivity k for the sender, or a single receiver is naive with probability α, for any $k > 0$ or $\alpha > 0$. Here is the intuition for why a sender, although upward biased, would not want to deviate from the separating equilibrium strategy μ. Suppose first that lying is costly (and the receiver is strategic). Since in equilibrium, the receiver inverts message m and believes that the state is in fact $x = \mu^{-1}(m)$, this gives the sender of type x an incentive to send a message $m > \mu(x)$. But as long as $\mu(x) \geq x$, this entails an increment in the cost for lying $C(m, x)$. For any

[20] Simple modifications of our arguments can be used to show that for any $m_0 > m^*(\underline{x})$, there is an increasing, differentiable separating equilibrium with $\mu(\underline{x}) = m_0$. If $m_0 < m^*(\underline{x})$, existence cannot be guaranteed without further assumptions, and, moreover, the equilibrium would be decreasing, which we find somewhat implausible when m^* is increasing.

[21] There are differences beyond just the state space, however. For example, his single-crossing condition in $\hat{x}$ and m is not satisfied in our setting. This makes our proofs of incentive compatibility distinct.

[22] While the differential equation (DE) is not Lipschitz around $(\underline{x}, m^*(\underline{x}))$, the inverse differential equation is.

$k > 0$, since the costs for lying are strictly convex in m, it is possible to find $\mu(x)$ large enough that the increment in the lying cost makes up for the gain from cheating the receiver.

When lying is costless, but the receiver is naive with probability α, the cost of inflating the equilibrium message $\mu(x)$ is induced by the response of the naive receiver. She will be persuaded, at least partially, by any message m above $\mu(x)$ and end up damaging the sender, as long as $y^R(\mu(\chi(x)))$ is larger than $y^S(x,b)$. For any $\alpha > 0$, since the sender's utility is strictly concave in y, it is possible to find $\mu(x)$ sufficiently large that the gain for cheating the strategic receiver is counteracted by the loss induced by the naive receiver.

As an illustration of both Theorem 2 and the above discussion, the following example analytically computes the Riley outcome for a quadratic-loss communication model with lying costs.[23]

Example 2. Suppose that the state space is $X = [0, \infty)$, lying is costly, and the receiver is fully strategic. Starting from the quadratic CS specification $U^R(y,x) = -(y-x)^2$ and $U^S(y,x,b) = -(y-(x+b))^2$, we add to the sender's payoff a lying cost of magnitude $-k(m-x)^2$ when message m is sent and the true state is x. Hence, the sender's utility is $U(x,\hat{x},m) = -(\hat{x}-x-b)^2 - k(m-x)^2$ when he sends message m and the receiver believes that the state is $\hat{x}$. A differentiable onto strategy μ is a separating equilibrium if and only if $\mu(x) \in max_m U\left(x, \mu^{-1}(m), m\right)$. By taking a first-order condition and substituting $\mu^{-1}(m)$ with x, and $\left(\mu^{-1}(m)\right)'$ with $\mu'(x)$, we obtain the differential equation

$$b - \mu'(x)\, k\, (\mu(x) - x) = 0, \tag{4}$$

with the initial condition for the Riley outcome being $\mu(0) = m^*(0) = 0$. Letting $v(x) = \mu(x) - x$ and $c = b/k$, so that $v'(x) = \mu'(x) - 1$, a rearranging of (4) yields

$$\frac{dv}{dx} = \frac{c-v}{v}. \tag{5}$$

Eq. (5) can be solved by integrating the further rearranged form, $[\nu/(c-v)]\, dv = dx$, to obtain the expression $-v - c\ln(c-v) = x + K$, where K is a constant to be determined. Substituting in $v = \mu - x$ gives the expression: $-(\mu(x)-x) - c\ln(c-(\mu(x)-x)) = x + K$. The initial condition $\mu(0) = 0$ implies that $K = -c\ln c$, and replacing c with b/k, we obtain the solution

$$-(\mu(x)-x) - \frac{b}{k}\ln\left(\frac{b}{k} - (\mu(x)-x)\right) = x - \frac{b}{k}\ln\frac{b}{k}. \tag{6}$$

It is straightforward to verify that μ defined by (6) satisfies $\mu(x) \in (x, x+\frac{b}{k})$ for all x (hence the logarithm in (6) is well-defined). Note that as $x \to \infty$, $\mu(x) \to x + \frac{b}{k}$. Thus, the strategy is approximately

[23] The calculations of the Riley outcome for the quadratic loss model of communication with a possibly naive receiver are very similar.

linear for most of the state space. In this example, there are no off-the-equilibrium path messages because $m^*(0) = 0$. □

3.3 Communication to a single receiver: lower unbounded state space

We now consider the case where the state space is unbounded below, i.e. $\underline{x} = -\infty$ and $X = (-\infty, \infty)$. The analysis builds on the results derived in Theorem 2. However, unlike when $\underline{x} > \infty$, one cannot pin down an initial condition such as $\mu(\underline{x}) = m^*(\underline{x})$ for the solution to (DE). Hence, there is an added complication of making sure that any local solution to (DE) can be extended leftward to $-\infty$. Intuitively, the difficulty is that while extending a local solution to the left, the solution might hit m^*, whereafter no further extension is possible since $-\frac{U_2}{U_3}$ (the right hand side of (DE)) is undefined at such a point. For instance, if it is the case that the slope of any increasing separating function, as given by (DE), is always larger and bounded away from the slope of the m^* function, then extension to $-\infty$ is impossible.[24] To surmount this problem, we need to impose more structure on the sender's utility.

For expositional clarity, we focus directly on a restricted class of problems that nevertheless capture a range of economic applications. Recalling the separable representation (2) of the sender's utility, we impose that the two components $U^S(y, x, b)$ and $G(m, x, b)$ always have the same shape regardless of the state of the world.

Definition 2. *The utility functions $U^S(y, x, b)$ and $G(y, x, b)$ are* shape invariant *if they can be represented as $U^S(y, x, b) = L\left(y - y^S(x, b)\right)$ and $G(y, x, b) = l(y - y^*(x, b))$, where the loss functions L and l are $\mathcal{C}^2$, $L'' < 0$, $l'' < 0$, $L'(0) = 0$, and $l'(0) = 0$. The function $l(\cdot)$ is assumed to have unbounded derivatives, and it is assumed that there exist constants θ_1 and θ_2 such that for all x, ${y^R}'(x) < \theta_1 < \infty$ and $0 < y^S(x, b) - y^R(x) < \theta_2$.*

While this class of utility functions is restricted within our general setting, it allows for asymmetric losses around the sender's bliss point, it is flexible with respect to risk aversion, and it allows the magnitude of the sender's bias to change with the state of the world. In particular, it accommodates our Applications 2 and 3 with a wide range of CS utility functions U^S and U^R, including the commonly used specifications in the literature.

The following theorem says that within the shape invariant class, there exists a separating equilibrium even when the state space is unbounded below.

[24] Proposition 1 in the Appendix of the working paper version of this article formally proves this.

Theorem 3. *Suppose the state space is* $(-\infty, +\infty)$ *and* $U(x, \hat{x}, m)$ *can be represented as in (2), with the functions* $U^S(y, x, b)$ *and* $G(y, x, b)$ *being shape invariant. Then there exists a strictly increasing differentiable separating equilibrium where* μ *solves (DE), is onto, and* $\mu(x) > m^*(x)$ *for all* x*.*

We conclude this section by reporting an example of information transmission to a single receiver who may be fully naive, with the state space being unbounded both above and below.

Example 3. Suppose that the state space is $X = (-\infty, \infty)$, lying is costless but the receiver may be fully naive with probability $\alpha > 0$, and utilities are CS quadratic loss functions. Specifically, $U^R(y, x) = -(y - x)^2$ and $U^S(y, x, b) = -(y - (x + b))^2$. Suppose that in equilibrium, the sender adopts a differentiable invertible function μ as her communication strategy. When message m is sent, a strategic receiver correctly infers the state $\mu^{-1}(m)$ and a naive receiver plays the action $y^R(m) = m$. The sender's expected utility from sending message m is thus

$$U\left(x, \mu^{-1}(m), m\right) = -(1-\alpha)\left(\mu^{-1}(m) - x - b\right)^2 - \alpha(m - x - b)^2.$$

By taking the first-order condition and substituting $\mu^{-1}(m)$ with x, and $\left(\mu^{-1}(m)\right)'$ with $\mu'(x)$, we obtain the necessary condition

$$(1-\alpha)b - \alpha(\mu(x) - x - b)\mu'(x) = 0. \tag{7}$$

This differential equation has a linear solution

$$\mu(x) = x + \frac{b}{\alpha},$$

which defines a differentiable separating equilibrium where the sender's strategy is onto.

Although this is the same equilibrium strategy that we identified in Example 1, an important difference is that in the current setting, $m^*(x) = b < \mu(x)$. Consequently, while it is also true here that in this equilibrium, $\mu(x)$ increases pointwise in b and decreases pointwise in α, and that $\mu(x) \to \infty$ pointwise as $\alpha \to 0$, the welfare implications for the sender are quite different. Recall that in Example 1, describing an instance of communication to a pool of receivers, the sender's utility did not change with α, because in equilibrium, the average action elicited was always her bliss point, $y^S(x, b)$. On the other hand, the sender's utility in the current equilibrium is given by

$$U(x, x, \mu(x)) = -(1-\alpha)b^2 - \alpha\left(\frac{b}{\alpha} - b\right)^2 = -b^2\left(\frac{1}{\alpha} - 1\right).$$

Hence the sender achieves her bliss point only when $\alpha = 1$, her utility strictly increases in α, and diverges to minus infinity as $\alpha \to 0$.[25] □

[25] In general, there will also exist partial-pooling equilibria which may give the sender higher expected utility but the strategic receiver a lower level of expected utility than the separating equilibrium here; see Ottaviani and Squintani [23].

4 Related literature

It is worth comparing our findings with those obtained by Morgan and Stocken [21] in a model of cheap talk where the sender has private information not only about the state (with bounded support, as in CS) but also about his bias, which can be either a positive, state-independent constant or zero.[26] When the possible bias is low, their game admits a partially separating equilibrium in which low states are fully revealed, while high states are partitioned in bunches.[27] But unlike in our model, there is no fully-separating equilibrium in the unbounded state space version of their model. Even though average recommendations are always unbiased in equilibrium by the law of iterated expectation, Morgan and Stocken [21] show that in a partitional equilibrium with two messages, the recommendations given by the biased sender are (stochastically) higher than those of the unbiased sender. Hence, both their model and ours can explain why biased analysts tend to be more optimistic than unbiased analysts. Our model can explain also why average recommendations are overly optimistic.

Our work is most closely related to Kartik [15], Chen [6], and Blanes [4]. Kartik [15] studies the general bounded state space CS setting, but with messages that entail some cost of misreporting for the sender. He shows that any sequence of monotone equilibria of his model can only converge to the most-informative equilibrium of CS, under a standard regularity condition on preferences. Moreover, in the cheap talk extension (Manelli [19])—when both costless and costly messages are available—Kartik [15] proves existence in, and fully characterizes a class of, forward-induction equilibria, demonstrating that they feature inflated communication. Chen [6] also studies the bounded support case of CS, but restricts attention to the uniform prior with quadratic loss functions setup, adding a fraction of honest senders who are always truthful and a fraction of naive receivers who always blindly believe the sender. Chen [6] proves existence and uniqueness of monotonic equilibrium in the cheap talk extension of her game, and shows that the equilibrium converges to the most-informative CS equilibrium as the fraction of naive players becomes small. The most important difference between our work and that of Chen [6]

[26] Sobel [25] formulates the first fully-fledged dynamic model of communication by a sender with unknown motives: the receiver does not know whether the sender has identical or opposed preferences. In the early periods, the sender with opposed preferences builds credibility by reporting the state of the world truthfully, only to exploit this reputation in the late periods. In Benabou and Laroque's [1] model, the sender may be honest and truthfully reveal her noisy signal with positive probability. Consequently, dishonest senders are able to manipulate receivers through misleading announcements. Olszewski [22] studies a communication model in which players are fully strategic, but the sender is motivated by both the receiver's action and her own reputation as an honest (i.e., truthtelling) sender. When the latter component sufficiently dominates the former, he shows that truthful communication is the unique equilibrium so long as the receiver observes a signal that is informative about the sender's signal.

[27] Intuitively, the unbiased sender can credibly convey unfavorable information, because the biased sender has no interest in admitting that the state is low. The equilibrium must be partitional when the state is sufficiently high relative to the bias, since otherwise a biased sender who observes a low state would have an incentive to pretend that the state is high.

and Kartik [15] is that we assume an unbounded state space; consequently, separating equilibria exist in our model but not in theirs; moreover, we are able to work with a more general payoff specification. Blanes [4] studies communication from a possibly truthful sender to a fully strategic receiver when the state is normally distributed. Following a higher message, the receiver then believes that the sender is less likely to be truthful. The resulting equilibrium characterized by Blanes [4] is invertible only for states below a certain threshold.

On the technical side, parts of our analysis owe much to the techniques introduced by Mailath [17] to study separation in signaling games with a continuum of types. Indeed, Theorem 2 in this paper can be considered an extension of his analysis from a bounded type space to a type space that is unbounded above, albeit with some differences in assumptions. However, there is no analog in Mailath [17] for our Theorem 1, which covers communication to a pool of partially naive receivers, nor for our Theorem 3, which identifies a condition to extend Mailath's results to lower unbounded type spaces.

5 Conclusion

Our costly talk model subsumes at least three modifications of the classic cheap talk model of CS: communication may be to a pool of receivers, some of whom are partially naive; a single receiver may be either fully strategic or partially naive; or, misreporting may be costly to the sender due to inherent psychological costs, technological constraints, or auditing penalties. In these cases, we show that equilibrium communication may be inflated but still reveal precise information to strategic receivers, while deceiving naive receivers.

Certainly, the existence of separating equilibria at our full level of generality relies on the assumption that the message and state space for the sender are unbounded either above, or both above and below. We believe that this may be a reasonable assumption in many applications, as discussed in the introduction. On the other hand, in some applications it is natural to assume that the state space is bounded: for example, when a military expert reports on how to allocate a fixed sum of available funds between the air force and the army. We refer to Kartik [15] and Ottaviani and Squintani [23] for analyses of costly talk in a bounded state space. In these models, separating equilibria can fail to exist because the messages required to support separation are inflated (as in our analysis here), and the upper bound on the message space becomes a binding constraint. Thus, there is an interval of the highest types that must pool in equilibrium. Nevertheless, as the size of the state space increases, the size of the interval of types that pool vanishes relative to the size of the state space. To this extent, the underlying theme of our analysis carries over.

Throughout the paper, we have restricted our attention to separating equilibria. While it is intractable to solve for all (perfect Bayesian) equilibria, we are aware that non-separating equilibria can exist, at least in some special cases.[28] For example, Ottaviani and Squintani [23] show that "partitional" outcomes à la CS can survive when the fraction of naive receivers is sufficiently small, but that such equilibria are not completely satisfying because they rely on somewhat implausible out-of-equilibrium belief assignments. Rather than formally pursuing refinements, in this paper we have simply followed the common practice of focusing on separating equilibria in costly signaling games when they exist. However, the arguments in Kartik [15] suggest that a refinement such as *monotone D1 equilibrium* (Bernheim and Severinov [2]) will isolate separating equilibria in our model. It is also worth noting that we have confined our attention to differentiable equilibria. Insofar as our goal was to prove the existence of separating equilibria, this is not troubling.

We believe that the analysis of costly talk and the application to naive receivers may present exciting avenues for future research. Welfare implications of information transmission policies in the presence of both naive and sophisticated receivers are potentially important. Moreover, it may be insightful to study further specialized but richer models that allow for various degrees of naivete on the part of different receivers.

Acknowledgments

We thank Sandeep Baliga, Marco Battaglini, Thomas de Garidel-Thoron, Harrison Hong, John Morgan, Wojciech Olszewski, Werner Ploberger, Daniel Seidmann, Peter Norman Sørensen, and especially Joel Sobel, two anonymous referees, and the Associate Editor for very useful suggestions and justified incredulity. We alone are responsible for any lies that remain in this paper.

Appendix

Proof of Lemma 1. To ease notational burden, let $f(x, \hat{x}, m) \equiv (1-\alpha)\, y^R(\hat{x}) + \alpha y^R(\chi(m))$.

(A.1) is trivial. The solution m^* to $U_3(x,x,m) = U_1^S\left(f(x,x,m), x, b\right) \alpha\, {y^R}'(\chi(m))\chi'(m) = 0$ is such that $(1-\alpha)\, y^R(x) + \alpha y^R(\chi(m^*(x))) = y^S(x,b)$. Hence, m^* is well-defined by $m^*(x) = \chi^{-1}\left({y^R}^{-1}\left(\frac{y^S(x,b)-(1-\alpha)y^R(x)}{\alpha}\right)\right)$. Since $U_{11}^S < 0$ and $U_1^S(f(x,x,m^*(x)), x, b) = 0$, we have

$$U_{33}(x,x,m^*(x)) = U_{11}^S\left(f(x,x,m^*(x)), x, b\right) \alpha \left({y^R}'(\chi(m^*(x)))\chi'(m^*(x))\right)^2 < 0,$$

[28]This discussion is implicitly restricted to the three applications of Section 2, where the sender's utility is well defined for any belief the receivers possess, and not just for beliefs that put mass on a single type.

which verifies (A.2).

For (A.4), we compute $U_{12}(x,\hat{x},m) = U^S_{12}\left(f(x,\hat{x},m),x,b\right)(1-\alpha)\,y^{R\prime}(\hat{x}) > 0$, because $U^S_{12} > 0$ and $y^{R\prime} > 0$. For (A.5), we compute $U_{13}(x,\hat{x},m) = U^S_{12}\left(f(x,\hat{x},m),x,b\right)\alpha\, y^{R\prime}(\chi(m))\chi'(m) > 0$, because $U^S_{12} > 0$, $y^{R\prime} > 0$, and $\chi' > 0$.

The first two parts of (A.3) are satisfied because $y^{R\prime}$ and χ' are bounded away from ∞, hence $m^{*\prime}$ is bounded away from 0. The last part of (A.3) follows because for any x and $m < m^*(\underline{x})$,

$$\begin{aligned} U\left(x,\underline{x},m^*(\underline{x})\right) &= U^S\left(y^S(\underline{x},b),x,b\right) \\ &> U^S\left((1-\alpha)\,y^R(\underline{x}) + \alpha y^R(\chi(m)),x,b\right) \\ &= U(x,\underline{x},m), \end{aligned}$$

where the first equality is by definition of $m^*(\underline{x})$ and the inequality holds because $U^S_{11} < 0$ and $y^S(x,b) \geq y^S(\underline{x},b) > y^R(\underline{x}) > y^R(\chi(m))$.

(B.2) is satisfied because $y^{R^{-1}}$ and χ^{-1} are strictly increasing. Finally, for (B.1), we see that for any (x,m), because $y^{R\prime} > 0$, and $\chi' > 0$,

$$\begin{aligned} & sign\left[U_2(x,x,m)\right] \equiv sign\left[U^S_1\left(f(x,x,m),x,b\right)(1-\alpha)\,y^{R\prime}(x)\right] \\ = \;& sign\left[U^S_1\left(f(x,x,m),x,b\right)(1-\alpha)\,\alpha\, y^{R\prime}(\chi(m))\chi'(m)\right] \equiv sign\left[U_3(x,x,m)\right]. \end{aligned}$$

□

Proof of Lemma 2. (A.1) is trivial. Observe that $U_3(x,x,m) = (\alpha+k)\,G_1\left(g(m),x,b\right)g'(m)$; since $g' > 0$ and $G_1\left(y^*(x),x,b\right) = 0$, it follows that $m^*(x) = g^{-1}\left(y^*(x,b)\right)$ is well-defined. Substituting $y^*(x,b) = g\left(m^*(x)\right)$, and noting that $G_1\left(y^*(x,b),x,b\right) = 0$, we have

$$U_{33}(x,x,m^*(x)) = (\alpha+k)\,G_{11}\left(y^*(x),x,b\right)\left(g'\left(m^*(x)\right)\right)^2 < 0,$$

because $G_{11} < 0$. This verifies (A.2).

For (A.3), note first that $m^*(x) = g^{-1}\left(y^*(x,b)\right)$ is strictly increasing because g and y^* are strictly increasing in x. Since utility is separable in $\hat{x}$ and m, the third part of (A.3) follows. Next, we see that $m^{*\prime}(x) = g^{-1\prime}\left(y^*(x,b)\right)y^*_1(x,b)$; since y^*_1 is bounded away from 0 and g' is bounded away from infinity, it follows that m^* is unbounded above and also unbounded below when $\underline{x} = -\infty$.

For (A.4), we compute $U_{12}(x,\hat{x},m) = (1-\alpha)\,U^S_{12}\left(y^R(\hat{x}),x,b\right)y^{R\prime}(\hat{x}) > 0$ because $U^S_{12} > 0$ and $y^{R\prime} > 0$. For (A.5), we compute $U_{13}(x,\hat{x},m) = (\alpha+k)\,G_{12}\left(g(m),x,b\right)g'(m) > 0$ because $g' > 0$ and $G_{12} > 0$. For (C.1), we compute $U_2(x,x,m) = (1-\alpha)\,U^S_1\left(y^R(x),x,b\right)y^{R\prime}(x) > 0$, because $y^{R\prime} > 0$, $U^S_{11} < 0$ and $y^R(x) < y^S(x,b)$.

For (C.2), first observe that $U_2(x,x,m) = (1-\alpha)\,U^S_1\left(y^R(x),x,b\right)y^{R\prime}(x)$ is bounded away from ∞ by assumption. For part (ii) of (C.2), fix any x. There exists $\varepsilon > 0$ such that $U_{33}(x,x,m) \geq 0$ only if $|m - m^*(x)| > \varepsilon$. So it suffices to prove that $|U_3(x,x,m)|$ is bounded away from 0 on the domain $|m - m^*(x)| > \varepsilon$. By definition of $m^*(x)$ and the assumption that $G_{11} < 0$, it follows that $|G_1(g(m),x,b)|$ is bounded away from 0 on the domain $|m - m^*(x)| > \varepsilon$. Moreover, g' is bounded away from 0. Thus, $|U_3(x,x,m)| = (\alpha+k)\,|G_1(g(m),x,b)|\,g'(m)$ is indeed bounded away from 0 on the relevant domain. □

Proof of Theorem 1. Let μ be any differentiable separating equilibrium strategy. It is necessary (but not sufficient) that for any $x \in X$, $\mu(x) \in \arg\max_{m \in \mu(X)} U(x, \mu^{-1}(m), m)$. As noted in the text, the first-order condition for optimality is (FOC). It is clear that one solution to (FOC) is $\mu(x) = m^*(x)$, since, in this case, $U_2(x, x, \mu(x)) = U_3(x, x, \mu(x)) = 0$. This is the unique non-decreasing solution because for any $m \neq m^*(x)$, $\frac{U_2(x,x,m)}{U_3(x,x,m)}$ is well-defined and strictly positive by conditions (A.2) and (B.1); hence if there is some x with $\mu(x) \neq m^*(x)$, $\mu'(x) < 0$ by (FOC). Henceforth, fix $\mu(x) = m^*(x)$; this is separating by (B.2) and differentiable. For any $m \in \mu(X)$, let the equilibrium beliefs be $\beta(m) = \mu^{-1}(m)$. If $\underline{x} = -\infty$, then $\mu(X) = M$ by (A.3). If $\underline{x} > -\infty$, then for any $m < \mu(\underline{x})$, let $\beta(m) = \underline{x}$. To prove that this construction is a separating equilibrium, it needs to be shown that for all x, $\mu(x) \in \arg\max_{m \in M} U(x, \beta(m), m)$. Observe that when $\underline{x} > -\infty$, condition (A.3) implies that for any x and $m < \mu(\underline{x})$, $U(x, \beta(\mu(\underline{x})), \mu(\underline{x})) \geq U(x, \beta(m), m))$. Therefore, regardless of whether $\underline{x} > -\infty$ or $\underline{x} = -\infty$, it suffices to show that $\mu(x) \in \arg\max_{m \in \mu(X)} U(x, \beta(m), m)$. Fix any state $\tilde{x}$. Let $\underline{M} \equiv \mu(X) \cap (-\infty, \mu(\tilde{x})]$ and $\overline{M} \equiv \mu(X) \cap [\mu(\tilde{x}), \infty)$.

We first argue that $\mu(\tilde{x}) \in \arg\max_{m \in \underline{M}} U(\tilde{x}, \beta(m), m)$. Suppose not, towards contradiction. Then, since $U(\tilde{x}, \beta(m), m)$ is $\mathcal{C}^1$ in m, there exists some $\hat{m} \in \underline{M} \setminus \mu(\tilde{x})$ such that $\left.\frac{dU(\tilde{x},\beta(m),m)}{dm}\right|_{m=\hat{m}} < 0$. Letting $\hat{x} \equiv \beta(\hat{m}) < \tilde{x}$, we have

$$U_2(\tilde{x}, \hat{x}, \hat{m})\beta'(\hat{m}) + U_3(\tilde{x}, \hat{x}, \hat{m}) < 0 = U_2(\hat{x}, \hat{x}, \hat{m})\beta'(\hat{m}) + U_3(\hat{x}, \hat{x}, \hat{m}). \tag{8}$$

Conditions (A.4) and (A.5), together with $\beta' > 0$, imply that $U_2(x, \hat{x}, \hat{m})\beta'(\hat{m}) + U_3(x, \hat{x}, \hat{m})$ is strictly increasing in x, contradicting condition (8).

The argument for $\mu(\tilde{x}) \in \arg\max_{m \in \bar{M}} U(\tilde{x}, \beta(m), m)$ is similar. □

Lemma 3. *If $\mu : X \to M$ solves (DE), $\mu(\underline{x}) = m^*(\underline{x})$, and is incentive compatible, then μ is strictly increasing.*

Proof. The argument is that of Mailath's [17] Theorem 2, hence the proof is omitted. Details are available in the working paper version of this article. □

Lemma 4. *There is a unique solution on X to the restricted initial value problem: (DE), $\mu(\underline{x}) = m^*(\underline{x})$, and $\frac{d\mu}{dx} > 0$.*

Proof. The proof is similar to Kartik's [15] Lemma 5, which builds on Mailath's [17] Proposition 5. For convenience, denote $m_0 \equiv m^*(\underline{x})$.

Step 1: (Local uniqueness) Let $m_0 \equiv m^*(\underline{x})$. Consider the inverse initial value problem to find $\tau(m)$ such that

$$\tau' = \Gamma(m, \tau) \equiv -\frac{U_3(\tau, \tau, m)}{U_2(\tau, \tau, m)}, \quad \tau(m_0) = \underline{x}. \tag{9}$$

By (A.1) and (C.1), Γ is continuous and Lipschitz in a neighborhood of $(m_0, \underline{x})$. Hence, standard existence theorems (e.g., Coddington and Levinson [7], Theorem 2.3) imply that there is a unique location solution, $\tilde{\tau}$, to (9) on $[m_0, m_0 + \delta)$, for some $\delta > 0$; $\tilde{\tau} \in \mathcal{C}^1([m_0, m_0 + \delta))$. Since $\Gamma(m_0, \underline{x}) = 0$ while ${m^*}^{-1}$ has strictly positive derivative, δ can be chosen small enough such that for all $m \in (m_0, m_0 + \delta)$, $m > m^*(\tilde{\tau}(m))$ and thus $\tilde{\tau}'(m) > 0$. Defining $\tilde{\mu} \equiv \tilde{\tau}^{-1}$ gives a solution to the restricted initial value problem on $[\underline{x}, \tilde{x})$, for some $\tilde{x} > \underline{x}$; $\tilde{\mu} \in \mathcal{C}^1([\underline{x}, \tilde{x}))$ with $\tilde{\mu}' > 0$. Since the inverse of any (local) solution to the restricted initial value problem is a (local) solution to the inverse initial value problem, we have (local) uniqueness of a solution to the restricted initial value problem.

Step 2: (Unique extension) To prove that there is a unique extension of $\tilde{\mu}$ from $[\underline{x}, \tilde{x})$ to $[\underline{x}, \infty)$, it is sufficient to prove the following inductive step: if $\delta \in (\underline{x}, \infty)$ and $\tilde{\mu}$ is a solution on $[\underline{x}, \delta)$ that is $\mathcal{C}^1$ with $\frac{d\mu}{dx} > 0$, then there is a unique extension of $\tilde{\mu}$ to $[\underline{x}, \delta + \theta)$ for some $\theta > 0$, while maintaining $\frac{d\mu}{dx} > 0$ and $\tilde{\mu} \in \mathcal{C}^1([\underline{x}, \delta + \theta))$. (This is sufficient because if $x^* < \infty$ is the supremum over all x such that $\tilde{\mu}$ can be extended to $[\underline{x}, x)$, then $\tilde{\mu}$ has an extension to $[\underline{x}, x^*)$, and by the inductive step, an extension to $[\underline{x}, x^* + \theta)$ for some $\theta > 0$, which contradicts the definition of x^*.)

It remains to prove the inductive step. Suppose $\delta \in (\underline{x}, \infty)$ and $\tilde{\mu}$ is a solution to (DE) on $[\underline{x}, \delta)$, with $\tilde{\mu} \in \mathcal{C}^1([\underline{x}, \delta))$ and $\tilde{\mu}' > 0$. Let $m_\delta \equiv \lim_{x \uparrow \delta} \tilde{\mu}(x)$.

First we show that $m_\delta > m^*(\delta)$. Suppose not, towards contradiction. Then $m_\delta = m^*(\delta)$ (because $\tilde{\mu}(x) > m^*(x)$ for all $x \in (\underline{x}, \delta)$) and $\lim_{x \uparrow \delta} \frac{d\mu(x)}{dx} = \infty$. Let $a \equiv \sup_{x \in [\underline{x}, \delta]} {m^*}'(x)$; (A.1) implies that $a < \infty$. Since $\tilde{\mu} \in \mathcal{C}^1([\underline{x}, \delta))$, there exists $\hat{x} < \delta$ such that $\frac{d\mu}{dx}(x) > a$ for all $x \in [\hat{x}, \delta)$. Pick $\omega > 0$ such that $\tilde{\mu}(\hat{x}) > m^*(\hat{x}) + \omega$. We have

$$\begin{aligned} m_\delta &= \tilde{\mu}(\hat{x}) + \lim_{x \uparrow \delta} \int_{\hat{x}}^{x} \frac{d\mu}{dx}(\xi) d\xi \\ &> m^*(\hat{x}) + \omega + \int_{\hat{x}}^{\delta} \frac{d\mu}{dx}(\xi) d\xi \\ &> m^*(\hat{x}) + \omega + \int_{\hat{x}}^{\delta} {m^*}'(\xi) d\xi \\ &= m^*(\delta) + \omega, \end{aligned}$$

which contradicts $m_\delta = m^*(\delta)$.

Next, we show that $m_\delta < \infty$. Suppose not, towards contradiction. Then $\lim_{x \uparrow \delta} \tilde{\mu}'(x) = \infty$. By (C.2.i), $U_2(x, x, m)$ is uniformly bounded away from ∞ on the domain $\{(x, m) : x \le \delta\}$. Consequently, $\lim_{x \uparrow \delta} |U_3(x, x, \tilde{\mu}(x))| = 0$. Since $m^*(\delta) < \infty$ whereas $\lim_{x \uparrow \delta} \tilde{\mu}(x) = \infty$, this contradicts the implication of (C.2.ii) that for any $\varepsilon > 0$, $|U_3(x, x, m)|$ is bounded away from 0 on the domain $\{(x, m) : x \le \delta \text{ and } m > m^*(x) + \varepsilon\}$.

Therefore, $m_\delta \in (m^*(\delta), \infty)$. By (A.1) and (A.2), $-\frac{U_2(x,x,\mu)}{U_3(x,x,\mu)}$ is continuous, Lipschitz, and bounded in a neighborhood of (δ, m_δ). Standard extension theorems (e.g., Coddington and Levinson [7],Theorem 4.1 and preceding discussion) imply that there is a unique extension of $\tilde{\mu}$ to $[\underline{x}, \delta + \theta)$ for some $\theta > 0$; this extension is $\mathcal{C}^1$. Since $\frac{d\mu}{dx}$ can never hit 0 and solve (DE), $\frac{d\mu}{dx} > 0$ on $[\underline{x}, \delta + \theta)$. □

Proof of Theorem 2. Lemmas 3 and 4 establish that there is a unique strategy for the sender that is a candidate for a differentiable separating equilibrium with $\mu(\underline{x}) = m^*(\underline{x})$. For the remainder of this proof, μ refers to this strategy. For any $m \ge m^*(\underline{x})$, beliefs are given by $\beta(m) = \mu^{-1}(m)$. Note that since μ is strictly increasing and solves (DE), it is immediate that $\mu(x) > m^*(x)$ for all x; since m^* is unbounded above (condition (A.3)), $\mu^{-1}(m)$ is thus well-defined for all $m \ge m^*(\underline{x})$. For all $m < m^*(\underline{x})$, set beliefs to $\beta(m) = \underline{x}$. Clearly, beliefs thus defined are consistent with Bayes rule. It remains only to show incentive compatibility of μ.

Fix any type $\tilde{x}$. First, we argue that $\mu(\tilde{x}) \in \arg\max_{m \in [\mu(\underline{x}), \mu(\tilde{x})]} U(x, \beta(m), m)$. Suppose not, towards contradiction. Then, since $U(\tilde{x}, \beta(m), m)$ is $\mathcal{C}^1$ in m, there exists some $\hat{m} \in [\mu(\underline{x}), \mu(\tilde{x}))$ such that $\left.\frac{dU(\tilde{x}, \beta(m), m)}{dm}\right|_{m=\hat{m}} < 0$. Letting $\hat{x} \equiv \beta(\hat{m}) < \tilde{x}$, we have

$$U_2(\tilde{x}, \hat{x}, \hat{m})\beta'(\hat{m}) + U_3(\tilde{x}, \hat{x}, \hat{m}) < 0 = U_2(\hat{x}, \hat{x}, \hat{m})\beta'(\hat{m}) + U_3(\hat{x}, \hat{x}, \hat{m}). \tag{10}$$

Conditions (A.4) and (A.5) combined with $\beta' > 0$ imply that $U_2(x, \hat{x}, \hat{m})\beta'(\hat{m}) + U_3(x, \hat{x}, \hat{m})$ is strictly increasing in x, contradicting (10).

An analogous argument shows that $\mu(\tilde{x}) \in \arg\max_{m \in [\mu(\tilde{x}), \infty)} U(x, \beta(m), m)$. Finally, we need to show that for all $m < \mu(\underline{x})$, $U(\tilde{x}, \tilde{x}, \mu(\tilde{x})) \geq U(\tilde{x}, \beta(m), m)$. To see this, observe that for any message $m < \underline{x}$,

$$\begin{aligned} U(\tilde{x}, \tilde{x}, \mu(\tilde{x})) &\geq U(\tilde{x}, \beta(\mu(\underline{x})), \mu(\underline{x})) = U(\tilde{x}, \underline{x}, m^*(\underline{x})) \\ &> U(\tilde{x}, \underline{x}, m) = U(\tilde{x}, \beta(m), m), \end{aligned}$$

where the first inequality holds because we already showed that type $\tilde{x}$ does not prefer to mimic type $\underline{x}$, the second inequality is by condition (A.3), and the two equalities follow from the construction of μ and β. □

Lemma 5. *Suppose $U(x, \hat{x}, m)$ can be represented as in (2), with the functions $U^S(y, x, b)$ and $G(y, x, b)$ being shape invariant. For any x_0, there exists $m_0 > m^*(x_0)$ such that there is a solution μ on $(-\infty, \infty)$ that satisfies (DE) and $\mu(x_0) = m_0$.*

Proof. Assume the hypothesis of the Lemma. The proof proceeds in two steps.

Step 1: We argue that there exists $\kappa > 0$ such that

$$g(m) \geq g(m^*(x)) + \kappa \implies \left(g'(m) \frac{U_2(x, x, m)}{|U_3(x, x, m)|} \leq g'(m^*(x))\, m^{*\prime}(x) \right). \tag{11}$$

To show this, observe that in the shape invariant utility domain,

$$\begin{aligned} U_2(x, \hat{x}, m) &= (1 - \alpha)\, L'\left(y^R(\hat{x}) - y^S(x)\right) y^{R\prime}(\hat{x}), \\ U_3(x, \hat{x}, m) &= (\alpha + k) l'\left(g(m) - y^*(x)\right) g'(m), \end{aligned}$$

and $m^*(x)$ satisfies $l'(g(m) - y^*(x))\, g'(m) = 0$, i.e., $g(m^*(x)) = y^*(x)$, or $m^*(x) = g^{-1}(y^*(x))$. Thus, it suffices to show that there is $\varepsilon > 0$ such that for all m, x,

$$g(m) \geq g(m^*(x)) + \varepsilon \implies \left(\frac{1-\alpha}{\alpha+k}\right) L'\left(y^R(x) - y^S(x)\right) y^{R\prime}(x) \leq g'(m^*(x))\, m^{*\prime}(x) |l'(g(m) - g(m^*(x)))|.$$

As long as $m > m^*(x)$, the right hand side of the above inequality is increasing in m; thus the above inequality is satisfied if it holds for $m = g^{-1}(y^*(x) + K)$, for some $K > 0$, so that $g(m) = y^*(x) + K = g(m^*(x)) + K$. Since $m^{*\prime}(x) = y^{*\prime}(x)/g'(m^*(x))$, it is sufficient that there exists a $K > 0$ such that for all x,

$$\left(\frac{1-\alpha}{\alpha+k}\right) L'\left(y^R(x) - y^S(x)\right) y^{R\prime}(x) \leq y^{*\prime}(x) |l'(K)|.$$

There is indeed such a $K > 0$ because (i) $y^{R\prime}(x) < \theta_1 < \infty$ and $\gamma_3 < y^{*\prime}(x)$ for all x; (ii) $0 < y^S(x) - y^R(x) < \theta_2$ for all x and hence $L'\left(y^R(x) - y^S(x)\right) \leq L'(-\theta_2) < \infty$ by the properties of L; and (iii) since l' is unbounded below, there exists $K > 0$ large enough such that $\left(\frac{1-\alpha}{\alpha+k}\right) L(-\theta_2) M\theta_1 \leq \gamma_3 |l'(K)|$.

Step 2: For any x_0, pick m_0 satisfying $g(m_0) \geq g(m^*(x_0)) + \kappa$, where κ is that identified in Step 1. Since g is strictly increasing, $m_0 > m^*(x_0)$. Then there is a local solution to (DE) around (x_0, m_0)—call it μ. Extension to $+\infty$ follows by the same argument as in the proof of Lemma 4. It remains to be proven that μ can be extended to $-\infty$. Since $U_3(x, x, m) < 0$ for all $m > m^*(x)$, it suffices to show that there is no $\hat{x} \in (-\infty, x_0)$ such that $\lim_{x \downarrow \hat{x}} \mu(x) = m^*(\hat{x})$. For this, it is sufficient if for all $x < x_0$, $g(\mu(x)) \geq g(m^*(x)) + \kappa$. To verify this property, observe that for all $x < x_0$,

$$
\begin{aligned}
g(\mu(x)) - g(m^*(x)) &= g(\mu(x_0)) - g(m^*(x_0)) - \int_x^{x_0} \left[g'(\mu(y))\mu'(y) - g'(m^*(y)) m^{*\prime}(y)\right] dy \\
&\geq \kappa - \int_x^{x_0} \left[g'(\mu(y)) \frac{U_2(y, y, \mu(y))}{|U_3(y, y, \mu(y))|} - g'(m^*(y)) m^{*\prime}(y)\right] dy \\
&\geq \kappa,
\end{aligned}
$$

where the first inequality uses $g(m_0) \geq g(m^*(x_0)) + \kappa$ and the second inequality uses the same fact combined with (11). ☐

Lemma 6. *For any (x_0, m_0) with $m_0 > m^*(x_0)$, if m_0 is sufficiently close to $m^*(x_0)$, the extension of the local solution to (DE) through (x_0, m_0) hits m^* for some $x < x_0$.*

Proof. Fix an $x_1 < x_0$ and a $\theta > 0$. We argue that by picking $m_0 \in (m^*(x_0), m^*(x_0) + \theta)$ sufficiently small, the solution through (x_0, m_0) will hit m^* at some $x \in S = [x_1, x_0]$. Let $a = \max_{x \in S} m^{*\prime}(x)$. This is positive and finite by the assumptions on U. Now, for any c, there exists $\varepsilon \in (0, \theta)$ s.t. if $x \in S$ and $m \in (m^*(x), m^*(x) + \varepsilon]$, then $-\frac{U_2(x,x,m)}{U_3(x,x,m)} > c$; this is by conditions (C.1) and (A.2). Consider in particular $c = \max\left\{\frac{\theta + m^*(x_0) - m^*(x_1)}{x_0 - x_1}, a\right\}$. By construction, the extension of the local solution through $(x_0, m^*(x_0) + \varepsilon)$, call it μ, has $\mu'(x) > a$, hence $\mu(x) \leq m^*(x) + \varepsilon$ over the domain of maximal extension intersecting S. Now suppose towards contradiction that μ can be extended to some $x \leq x_1$. Since by construction $\mu'(x) > \frac{\theta + m^*(x_0) - m^*(x_1)}{x_0 - x_1}$ for all $x \in S$, we have

$$
\begin{aligned}
\mu(x_1) &= \mu(x_0) - \int_{x_1}^{x_0} \mu'(x)\, dx \\
&< m^*(x_0) + \varepsilon - \int_{x_1}^{x_0} \frac{\theta + m^*(x_0) - m^*(x_1)}{x_0 - x_1} dx \\
&= \varepsilon - \theta + m^*(x_1) \\
&< m^*(x_1),
\end{aligned}
$$

which implies by the intermediate value theorem that there exists some $x \in (x_1, x_0)$ such that $\mu(x) = m^*(x)$, a contradiction with μ solving (DE) at x. ☐

Lemma 7. *For any x_0 and $m_0 < \tilde{m}_0$, the left-extension of the local solution to (DE) through (x_0, m_0) lies strictly below the left-extension of the local solution to (DE) through $(x_0, \tilde{m}_0)$ (over the smaller of the two domains of possible extension).*

Proof. Let μ and $\tilde{\mu}$ be the respective solutions for the two initial conditions. Given the continuity of solutions, it suffices to show that there is no $x < x_0$ such that $\mu(x) = \tilde{\mu}(x)$. Suppose such a point exists. Let $\hat{x}$ be the supremum over such points below x_0; that $m_0 < \tilde{m}_0$ implies that $\hat{x} < x_0$. Then by the local uniqueness of solutions at any (x, m) such that $m \geq m^*(x)$ (using the local uniqueness of an increasing solution to the inverse differential equation of (DE) if $m = m^*(x)$), $\mu(\hat{x}) = \tilde{\mu}(\hat{x})$ implies that $\mu(x) = \tilde{\mu}(x)$ for some $x \in (\hat{x}, x_0)$, contradicting the definition of $\hat{x}$. □

Proof of Theorem 3. Assume that U can be represented as in (2) and the functions U^S and G are shape invariant. By Lemma 5, there is a solution to (DE) on the entire domain passing through (x_0, m_0) for some m_0 sufficiently large. If the solution is onto, we are done. Suppose not, so that it has a horizontal asymptote. Let $\bar{m}_0$ be the infimum over $m > m^*(x_0)$ for which (i) a solution exists over the entire domain; and (ii) the solution is *not* onto. Let $\underline{m}_0$ be the supremum over $m > m^*(x_0)$ for which no solution exists on entire domain. Lemmas 6 and 7 imply that $\infty > \bar{m}_0 \geq \underline{m}_0 > m^*(x_0)$. If $\bar{m}_0 > \underline{m}_0$, then for any $m_0 \in (\underline{m}_0, \bar{m}_0)$, the local solution through (x_0, m_0) extends over the whole domain and is onto by the definitions of $\underline{m}_0$ and $\bar{m}_0$, and we are done. So assume henceforth that $\bar{m}_0 = \underline{m}_0$. The theorem is proved by showing that the local solution to (DE) through $(x_0, \underline{m}_0)$ extends to $-\infty$ and is onto. For any $m_0 > m^*(x_0)$, denote by $\mu(x, m_0)$ the value at x of the (maximally extended) solution to (DE) with the initial condition (x_0, m_0).

Step 1: The left-extension, given initial condition $(x_0, \underline{m}_0)$, extends to $-\infty$. Suppose not, towards contradiction. Then there exists some $\hat{x}$ such that $\mu(\hat{x}, \underline{m}_0) = m^*(\hat{x})$. Let $D_{\varepsilon,\theta} \equiv [\hat{x} + \varepsilon, x_0 + \theta] \times (\mu(\hat{x} + \varepsilon), \mu(x_0 + \theta))$. The right hand side of the (DE) is continuous and Lipschitz on $D_{\varepsilon,\theta}$ for small enough $\varepsilon, \theta > 0$ since $D_{\varepsilon,\theta}$ is a bounded space. By Birkhoff and Rota's [3] Theorem 2 and Corollary from Chapter 6, the function $\mu(x, m_0)$ is continuous in m_0 in a neighborhood of $\underline{m}_0$. Thus, by choosing arbitrarily small $m_0 > \underline{m}_0$, $\mu(\hat{x} + \varepsilon, m_0)$ can be made close to $\mu(\hat{x} + \varepsilon, \underline{m}_0)$, but strictly larger by the monotonicity Claim. Now, noting that by choosing small enough ε, $\mu(\hat{x} + \varepsilon, \underline{m}_0)$ can be made arbitrarily close to $m^*(\hat{x} + \varepsilon)$, it follows that by choosing small enough ε and small enough $m_0 > \underline{m}_0$, $\mu(\hat{x} + \varepsilon, m_0)$ can be made arbitrarily close to $m^*(\hat{x} + \varepsilon)$. But then, by the non-existence Claim, the solution $\mu(\cdot, m_0)$ cannot extend to $-\infty$ because it hits m^* for some $x < \hat{x} + \varepsilon$. This contradicts $m_0 > \underline{m}_0 = \bar{m}_0$ and the definition of $\bar{m}_0$.

Step 2: The left-extension, given initial condition $(x_0, \underline{m}_0)$, is onto. Suppose not, towards contradiction. Then by Step 1, $\mu(x, \underline{m}_0)$ has a horizontal asymptote as $x \to -\infty$—call it $\hat{m}$. Since $\mu(x, \underline{m}_0)$ is strictly increasing in x, $\mu(x, \underline{m}_0) > \hat{m}$ for all x. Consider some small $\theta > 0$. There exists $\varepsilon > 0$ such that for all $x \in (-\infty, x_0 + \theta]$, $\mu(x, \underline{m}_0) - m^*(x) > \varepsilon$, because $\lim_{x \to -\infty} m^*(x) = -\infty$. Define $E_{\varepsilon,\theta} \equiv \{(x, m) : x \in (-\infty, x_0 + \theta], m > m^*(x) + \varepsilon\}$. The right hand side of (DE) is continuous and Lipschitz on $E_{\varepsilon,\theta}$ for small enough $\varepsilon, \theta > 0$, because of condition (C.2). Applying again Birkhoff and Rota's [3] Theorem 2 and Corollary from Chapter 6, the function $\mu(x, m_0)$ is continuous in m_0 in a neighborhood of $\underline{m}_0$. Thus, for any $x_1 \in (-\infty, x_0)$, there exists large enough $m_0 < \underline{m}_0$ such that $\mu(x_1, m_0)$ is arbitrarily close to $\mu(x_1, \underline{m}_0)$, in particular $\mu(x_1, m_0) > \hat{m}$. Recalling the constant κ identified in condition (11), it follows that because $m^*(x)$ is unbounded below and g is strictly increasing with a derivative bounded away from 0, we can choose x_1 such that $g(\hat{m}) > g(m^*(x_1)) + \kappa$. But then, $\mu(x_1, m_0)$ extends to $-\infty$ by the argument of Lemma 5, contradicting the definition of $\underline{m}_0$. □

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