Observational Learning with Ordered States

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Motivation (1)

Sequential observational learning model (Banerjee '92; BHW '92)

- unknown state $\omega \in \Omega$
- each $n = 1, 2, \ldots$ takes action $a_n \in A$ (discrete set)
  using private signal and (full) history of actions
- homogenous prefs $u(a_n, \omega)$

Many extensions, variations

Fundamental Q: does society eventually learn ω?

Received A: Unbounded vs. bounded beliefs/signals (Smith & Sørensen '00)

- can private beliefs $\to$ certainty about every ω?
- are private beliefs bounded away from 0 about every ω?

Motivation (2)

Unbounded beliefs $\implies$ learning for **all** prefs

Bounded beliefs $\implies$ nonlearning for **all** (nontrival) prefs

Essentially exhaustive with two states (most papers)

But with multiple states, a large gap

Suppose $\Omega = \{1, 2, 3\}$ and signals $\mathcal{N}(\omega, 1)$

- Can become certain about 1 or 3 but not 2
- Neither unbounded nor bounded!

So is there learning? Say for $u(a, \omega) = -(a - \omega)^2$

This Paper

Prefs satisfying SCD: single-crossing differences (Milgrom & Shannon '94)

- widely used in economics; covers quadratic loss
- but not previously in social learning

Information satisfying DUB: directionally unbounded beliefs

- new property: **can approach certainty about each state vs. (only) lower/upper sets**
- weaker than unbounded beliefs; covers normal info

Main results

1. SCD & DUB are a *minimal pair* of sufficient conditions for learning
2. New perspective: learning turns on detecting suboptimal actions
   $\rightarrow$ not (directly) learning the optimal action / true state
3. General characterization of info diffusion + learning on networks

Literature

Most related (will elaborate later)

- Smith & Sørensen '00; Arieli & Mueller-Frank '21
- Acemoglu, Dahleh, Lobel, Ozdaglar '11; Lobel & Sadler '15

Other mechanisms for learning

- Suitably heterogenous preferences Goeree, Palfrey, Rogers '06
- Prices/congestion costs
  Avery & Zemsky '98; Eyster, Galeotti, Kartik, Rabin '14

Non-Bayesian / Misspecified learning

Model

Environment

Countable set of states $\Omega \subset \mathbb{R}$ $(|\Omega| \leq \infty)$

Measurable signal set S

- when MLRP is mentioned, S is totally ordered

Signal structure $f(s|\omega)$ (R-N densities)

- no signal can exclude any state: $f(\cdot) > 0$

Countable action set A $(|A| \leq \infty)$

Note: more general setup covered in paper, e.g., Ω and A uncountable

The Game

Unobservable state ω drawn from prior pmf $\mu_0 \in \Delta\Omega$

Agents $1, 2, \ldots$ sequentially choose actions; each agent n observes both

- conditionally indep private signal $s_n \sim f(\cdot|\omega)$
- actions of all predecessors in her neighborhood $B(n) \subseteq \{1, \ldots, n-1\}$

$B(\cdot)$ defines **social (observational) network** structure (common knowledge)

- notable cases: immediate predecessor; full history
- for talk, only deterministic networks; papers covers stochastic networks
  - $\rightarrow$ stochastic e.g.: single random predecessor

Strategy $\sigma_n : S \times A^{B(n)} \rightarrow \Delta A$

All agents share bounded vNM utility $u : A \times \Omega \rightarrow \mathbb{R}$ (assm optimal action exists $\forall$ beliefs)

Bayes Nash equilibria (or refinements)

$\rightarrow$ no real strategic interaction

Learning

Full-information exp utility $u^*(\mu) := \sum_\omega \max_a u(a,\omega)\mu(\omega)$

Given prior μ_0 and eqm σ, agent n has ex-ante exp utility $\mathbb{E}_{\sigma,\mu_0} u_n$

Definition

There is adequate learning if for every prior μ_0 and every eqm σ, $\mathbb{E}_{\sigma,\mu_0} u_n \to u^*(\mu_0)$ as $n \to \infty$.

Adequate learning impossible unless network satisfies:

Assumption

The social network has expanding observations: $\forall K \in \mathbb{N}$, $|\{n : B(n) \subseteq \{1,\dots,K\}\}| < \infty$.

Failure $\iff$ an infinite set of agents can observe only a finite set of actions

Under expanding obs, for what (u, f) is there adequate learning?

SCD Preferences

and

DUB Information

Single-Crossing Differences

$h : \mathbb{R} \to \mathbb{R}$ is single crossing if either

$$\forall x < x' \colon\ h(x) > 0 \implies h(x') \geq 0; \qquad \text{(upcrossing)}$$

or

$$\forall x < x' \colon\ h(x) < 0 \implies h(x') \leq 0. \qquad \text{(downcrossing)}$$

Definition

Utility $u : A \times \Omega \to \mathbb{R}$ has single-crossing differences (SCD) if

$$\forall a, a' :\ u(a,\omega) - u(a',\omega) \text{ is single crossing in } \omega.$$

- à la Milgrom & Shannon '94, but no order on A (Kartik, Lee, Rappoport '19)
- implied by supermodularity (wrt to some order on A)

Directionally Unbounded Beliefs

Given state ω and belief μ,

let $\mu(\omega|s)$ be posterior after signal s, and $\mu(\omega|\cdot)$ the corresponding r.v.

Say that ω is distinguishable from Ω' if $\forall \mu \in \Delta(\Omega' \cup \omega)$ with $\mu(\omega) > 0$, $\operatorname{Supp} \mu(\omega|\cdot) \ni 1$

$\rightarrow$ can approach relative certainty about ω vs. all of Ω', simultaneously

Definition

There is directionally unbounded beliefs (DUB) if every ω is distinguishable from $\{\omega' : \omega' < \omega\}$ and also from $\{\omega' : \omega' > \omega\}$.

Crucially, need not distinguish ω simultaneously from both lower and higher states

If Ω finite (and writing as if S countable), DUB $\iff$ $\forall \omega$:

1. $\exists\ (\overline{s}_i)$ s.t. $\forall \omega' < \omega$, $\lim_{i \to \infty} f(\overline{s}_i|\omega')/f(\overline{s}_i|\omega) = 0$;
2. $\exists\ (\underline{s}_i)$ s.t. $\forall \omega' > \omega$, $\lim_{i \to \infty} f(\underline{s}_i|\omega')/f(\underline{s}_i|\omega) = 0$.

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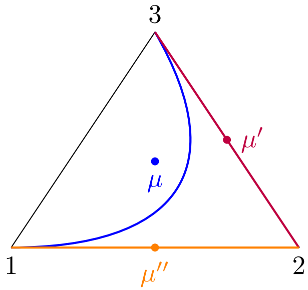
Normal info:
$f(s|\omega) = \omega + \mathcal{N}(0, 1)$



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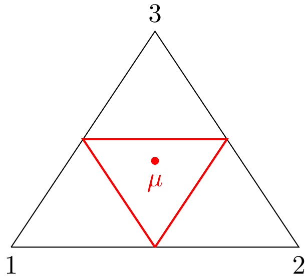
Violation



Main Results

Two Results on Learning

Recall that network satisfies expanding observations

Theorem (SCD & DUB)

SCD prefs & DUB info $\implies$ adeq learning. They are a minimally sufficient pair (varying A).

Let $I(\mu)$ be exp utility improvement from drawing a signal given belief μ

Say that μ with $I(\mu) = 0$ is stationary, as signals can then be ignored

Say that μ has adequate knowledge if $\exists a$ that is optimal $\forall \omega \in \mathrm{Supp}\,\mu$

Straightforward: adeq learning $\implies$ all stationary beliefs have adequate knowledge

$\rightarrow$ If $I(\mu_0) = 0$, there can be an immediate info cascade

Theorem (Backbone)

Adequate learning $\iff$ all stationary beliefs have adequate knowledge.

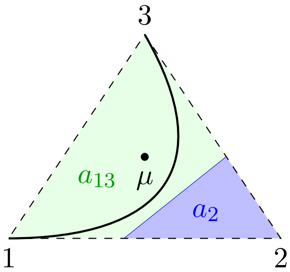
SCD-DUB Theorem

Theorem

SCD prefs & DUB info $\implies$ adeq learning.

Backbone Thm: do all stationary beliefs have adeq knowledge?

Not in general $\because$ "mismatch" btwn prefs and info



μ is stationary and has <u>in</u>adeq knowledge ... but SCD violated

SCD-DUB Theorem

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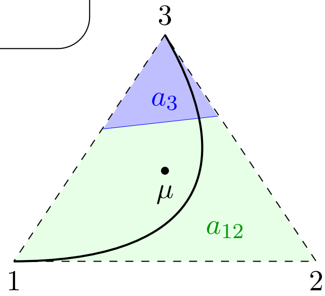
SCD and DUB together preclude mismatch

(faulty intuition)



Proof sketch:

SCD $\implies$ if μ has IAK then $\exists \omega^* \in \operatorname{Supp} \mu$ s.t.

$$u(a(\omega^*), \omega^*) > u(a(\mu), \omega^*) \quad \textbf{and}$$

$$u(a(\omega^*), \omega) \geq u(a(\mu), \omega) \quad \forall \omega > \omega^* \text{ or } \forall \omega < \omega^*$$

DUB $\implies$ $\exists$ signals that "rule out" both $\{\omega : \omega < \omega^*\}$ and $\{\omega : \omega > \omega^*\}$

So SCD + DUB $\implies$ any μ with IAK is not stationary

Nb: crucial is **ruling out** wrong a; not taking correct a

SCD-DUB Theorem

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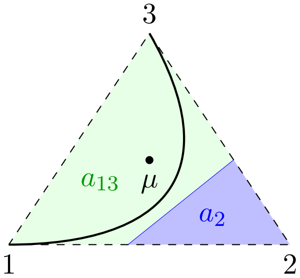
SCD and DUB together preclude mismatch

Absent SCD, there is DUB info with inadeq learning (for some action set):



In fact, given any non-SCD pref, any MLRP info $\implies$ inadeq learning (for some action set)

SCD-DUB Theorem

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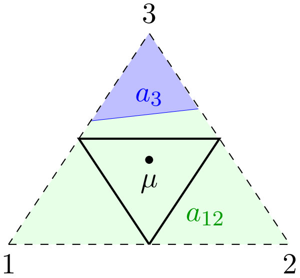

Here an upper set is not distinguishable from its complement

More generally, may need to use non-full-support prior

(But can still use a full-support prior given, e.g., MLRP)

Backbone Theorem

Theorem

Adequate learning $\iff$ all stationary beliefs have adequate knowledge.

Proof idea ($\Longleftarrow$): (elaborate)

1. If agent's social belief is not close to stationary, can achieve a min utility improvement
   - → work with social belief **distrs**; Bayes-Plausibility yields compactness (use Prohorov Thm)
   - → complement of ε-nbhd of stationary beliefs is a closed subset (so also compact)
   - → exp utility / improvement is cts in belief, also cts on distrs (weak topology)
2. Expanding observations $\implies$ improvement principle: these min improvements propogate (e.g., consider immediate-predecessor network); so they can occur only finitely often
   - → eventually as if every agent has arb. close to stationary social belief
   - → asymptotic exp utility is at least that of the worst stationary belief: "info diffusion"
   - → when all stationary beliefs have adeq knowlege, there is adeq learning

Backbone Theorem

Theorem

Adequate learning $\iff$ all stationary beliefs have adequate knowledge.

This is a general result, not just for SCD + DUB

Subsumes most existing learning results (and info diffusion; Lobel & Sadler '15)

- Including "responsive prefs" with infinite action spaces (Lee '93; Ali '18)
  - → E.g., if $\Omega = \{0, 1\}$, $A = \mathbb{Q}$, and $u(a, \omega) = -(a - \omega)^2$, then given any info structure (not uninformative), only stationary beliefs are $\{0, 1\}$
- Suppose only 2 states and finite actions, as much of the literature
  - → Adeq knowledge means knowing the state (modulo trivialities)
  - → So signals must be able to provide (almost-)certainty about each state: unbounded beliefs

Backbone Theorem

Theorem

Adequate learning $\iff$ all stationary beliefs have adequate knowledge.

Yields new perspective on what drives learning

- Info must be able to rule out all wrong actions
  - $\rightarrow$ even with SCD + DUB, could exist beliefs at which no signal leads to correct action; yet beliefs will eventually be driven to taking the correct action
- Given a pref u, a sufficient condition for learning is that info provides excludability:

  For any a, a', $\{\omega : u(a,\omega) > u(a',\omega)\}$ is distinguishable from $\{\omega : u(a',\omega) > u(a,\omega)\}$

  i.e., if a is better than a', must be able to learn that
- Excludability $\forall$ SCD prefs $\iff$ DUB
- Excludability $\forall$ prefs $\iff$ unbounded beliefs ... But very demanding for $|\Omega| > 2$

Proposition

Assume $|\Omega| > 2$. MLRP $\implies$ NOT unbounded beliefs.

Recall Normal info

Discussion

DUB in Location Families

Location family: $S = \mathbb{R}$ and for some density g, $f(s|\omega) = g(s - \omega)$

- e.g., Normal info
- Intuitively, DUB requires a thin tail of standard density g

g strictly subexponential: $\exists p > 1$ s.t. $g(x) < \exp[-|x|^p]$ for large $|x|$

Proposition

In a location family, DUB holds if g is strictly subexponential.

- If g is exponential then $g(s - \omega')/g(s - \omega) = \exp(\omega' - \omega)$ is indep of s
- An even thicker tail (superexp) makes extreme signals uninformative
- So Laplace & Cauchy distrs fail DUB

Most-Related Papers

Line network: Smith & Sørensen '00 (two states) and Arieli & Mueller-Frank '21 (general)

- unbounded beliefs characterizes learning for all prefs
- AMF '21: "vanishing value of private information", analogous to our Backbone Thm
  - → Martingale approach, which fails for general networks

General networks, but only **two states and two actions**

- Acemoglu, Dahleh, Lobel, Ozdaglar '11 introduce improvement principle
- Lobel & Sadler '15 introduce info diffusion (and correlated networks)
  - → Both rely critically on two states/actions to derive minimum improvement
  - → Our approach using compactness/continuity is novel and works generally

SCD + DUB are new, and more generally study of learning for broad pref classes

- AMF '21 show that *pairwise UB* is sufficent for a special utility

Conclusion

Conclusion

Std condition for learning, unbounded beliefs, very demanding with > 2 states

We tackle learning for canonical economic prefs with ordered states, SCD

New informational condition: DUB

$\rightarrow$ unlike unbounded beliefs, compatible with MLRP and satisfied by Normal info

Main results:

1. DUB info and SCD prefs are minimal pair of suff conditions for learning
   $\rightarrow$ in general social networks, given expanding observations
2. General theorem on info diffusion in networks (given expnd obsvns)
   $\rightarrow$ clean characterization of learning: all stationary beliefs have adeq knowledge

Interesting future directions:

- Other pairs of suff cond
- Heterogenous prefs
- Speed of convergence
- DUB in other contexts

Thank you!

Faulty Intuition for Sufficiency

(thm)

- Take μ with inadequate knowledge
  1. SCD implies different optimal actions at extreme states of $\operatorname{Supp}\mu$
  2. DUB implies potential certainty about extreme states of $\operatorname{Supp}\mu$
  3. μ is non-stationary

- Is learning about SCD or different optimal actions at extreme states?

- Above logic fails with infinite states, or $\because$ SCD holds only weakly

  $\rightarrow$ Not about distinct optimal actions

Faulty Intuition for Sufficiency

(thm)

- Let $\Omega = \mathbb{Z}$ and $A = \mathbb{Z} \cup \{a^*\}$

$$u(a, \omega) = \begin{cases} 1 & \text{if } a = \omega \\ 0 & \text{if } a \notin \{\omega, a^*\} \\ 1 - \varepsilon & \text{if } a = a^* \end{cases}$$

- For small $\varepsilon > 0$, a^* is a safe action but suboptimal in every state
- Different optimal action in every state, but u violates SCD
- Suppose $s \sim \mathcal{N}(\omega, 1)$. For any full support prior
  - signals cannot provide certainty about any state
  - for small enough $\varepsilon > 0$ the prior is stationary

More on Backbone Theorem

Theorem

Adequate learning $\iff$ all stationary beliefs have adequate knowledge.

Some elaboration ($\Longleftarrow$): (ideas)

Ex ante, each n has a Bayes-Plausible distr of "social" (interim) beliefs, $\varphi \in \Phi^{BP} \subset \Delta\Delta\Omega$

Φ^{BP} is compact ($\Delta\Delta\Omega$ metrized by Prohorov)

Let $\Phi^S \subseteq \Phi^{BP}$ be distrs supported on stationary beliefs; and let Φ^S_ε be an ε-nbhd

Expected improvement $I(\varphi)$ is cts, so attains minimum $\delta(\varepsilon) > 0$ over $(\Phi^S_\varepsilon)^c$ (closed hence compact)

Whereas for $\varphi \in D^\varepsilon$, $u(\varphi) > u^* - \gamma(\varepsilon)$, with $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$ (using unif cont of u)

By an improvement principle, $\liminf_n \mathbb{E}u_n \geq u^* - \gamma(\varepsilon)$ (this step adapts ADLO '11)

- E.g., consider immediate-predecessor network
- Each $\mathbb{E}u_n \geq \min\{u^* - \gamma(\varepsilon), \mathbb{E}u_{n-1} + \delta(\varepsilon)\}$
- Iterate

Since $\varepsilon > 0$ is arbitrary, $\liminf_n \mathbb{E}u_n \geq u^*$, which implies $\mathbb{E}u_n \to u^*$