A dimension descent scheme for the positive mass theorem in arbitrary dimension

Simon Brendle, Columbia University

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The classical positive mass theorem in dimension $n \leq 7$

Theorem (Schoen-Yau 1980s): Let $3 \leq n \leq 7$. Suppose that (M, g) is an asymptotically flat manifold of dimension n and $g_{ij} = (1 + \alpha\, r^{2-n})\, \delta_{ij} + O(r^{2-n-2\delta})$ near infinity. If g has strictly positive scalar curvature at each point on M, then the mass term α is positive.

The Schoen-Yau proof is based on a striking dimension descent scheme, and uses on minimal hypersurfaces. Specifically, if the mass term α is nonpositive, one can construct an area-minimizing hypersurface Σ in M. Since $n \leq 7$, Σ is free of singularities. Let $\hat{g}$ denote the induced metric on Σ. By a conformal change of the metric, one can construct a metric $\tilde{g}$ on Σ with the property that $\tilde{g}$ has strictly positive scalar curvature at each point on Σ, $\tilde{g}$ is asymptotically flat, and $\tilde{g}$ has zero mass.

The Schoen-Yau argument is one of the landmark achievements of modern differential geometry.

Previous work in dimension $n \geq 8$

In dimension $n \geq 8$, area minimizing hypersurfaces may have singularities. The singular set is known to have Hausdorff dimension at most $n - 8$.

Chodosh-Mantoulidis-Schulze-Wang showed that, for $8 \leq n \leq 11$, singularities can be avoided by slightly perturbing the metric on ambient space. Consequently, the positive mass theorem holds in those dimensions.

Bi-Hao-He-Shi-Zhu recently gave a proof of the positive mass theorem up to dimension 19. Their work relies on the conformal blow-up technique. Specifically, they modify the metric conformally, with a conformal factor that blows up at the singular set. The conformal factor is constructed by solving a linear PDE on a singular space.

Schoen-Yau (2017) and Lohkamp (2006) have proposed proofs of the positive mass theorem in all dimensions.

Setting up the inductive scheme

Definition: Let $n \geq 3$. An n-dataset consists of a complete Riemannian manifold (M, g) of dimension n together with positive smooth functions ρ and Q satisfying the following conditions:

- The manifold (M, g) has an asymptotically flat end E_0. The metric satisfies $g = (1 + \alpha\, r^{2-n})\, \bar{g} + O(r^{2-n-2\delta})$ on E_0, where $\bar{g}$ denotes the Euclidean metric on E_0 and α is a constant.
- If $n = 3$, we assume that (M, g) has no ends other than E_0. If $n \geq 4$, we allow (M, g) to have arbitrary ends in addition to the asymptotically flat end E_0.
- The function ρ satisfies $\rho = 1 + \beta\, r^{2-n} + O(r^{2-n-2\delta})$ on E_0, where β is a constant.
- The function Q satisfies $Q = O(r^{2-n-2\delta})$.
- We require corresponding estimates for the higher derivatives of g, ρ, and Q.

Setting up the inductive scheme (continued)

- We require that a certain quadratic form is nonnegative. Specifically, we assume that

$$\int_M \rho\,|df|^2 + \frac{1}{2}\int_M \rho\left(R - 2\,\Delta\log\rho - \frac{n+1}{n+2}\,|d\log\rho|^2\right) f^2 \geq \int_M \rho\,Q\,f^2$$

  for every smooth test function f with the property that the set $\{f \neq 0\} \setminus E_0$ is bounded and there exists a constant a such that the set $\{f \neq a\} \cap E_0$ is bounded.

Definition: Let $n \geq 3$, and let (M, g, ρ, Q) be an n-dataset. We define the mass of (M, g, ρ, Q) to be $(n-1)\alpha + 2\beta$.

Statement of the main theorem

Theorem (B.-Wang, April 2026): Let $n \geq 3$, and let (M, g, ρ, Q) be an n-dataset. Then the mass of (M, g, ρ, Q) is strictly positive, i.e. $(n-1)\alpha + 2\beta > 0$.

If we put $\rho = 1$ and $Q = \frac{1}{2} R$, the positive mass theorem follows.

The proof uses a number of ingredients. One of them is the shielding principle of Lesourd-Unger-Yau. We also use a conformal blow-up argument in the spirit of Bi-Hao-He-Shi-Zhu. Finally, the work of Cheeger and Naber on the Minkowski dimension of the singular set plays a crucial role in the proof.

Proof of the main theorem

The proof of the main theorem proceeds by induction on n.

For $n = 3$, the assertion can be reduced to the classical work of Schoen-Yau. More precisely, if (M, g, ρ, Q) is a 3-dataset with nonpositive mass, we can construct a conformal metric on (M, g) which has zero scalar curvature and is asymptotically flat with strictly negative mass. This contradicts the classical positive mass theorem of Schoen and Yau in dimension 3.

Suppose next that $n \geq 4$, and that the theorem holds for $(n-1)$-datasets. Our goal is to show that the theorem holds for n-datasets. To prove this, we argue by contradiction. Suppose that (M, g, ρ, Q) is an n-dataset with nonpositive mass, so that

$$(n-1)\alpha + 2\beta \leq 0,$$

where α denotes the coefficient in the asymptotic expansion of the metric g and β denotes the coefficient in the asymptotic expansion of ρ. The goal is to construct an $(n-1)$-dataset with zero mass.

Step 1: Truncating all ends of M, except for the asymptotically flat end

Lemma (cf. Lesourd-Unger-Yau): Let E_0 denote the asymptotically flat end of M. We can find an open, connected domain E with smooth boundary, a smooth function Φ defined on E, and a smooth function $\hat{Q}$ defined on E with the following properties:

- The closure of E_0 is contained in E.
- The complement $E \setminus E_0$ is a bounded subset of (M, g).
- $\Phi = 0$ at each point in E_0.
- $\Phi \leq 0$ at each point in E.
- $\Phi \to -\infty$ on the boundary ∂E.
- $Q + \frac{1}{2}\Phi^2 - 2\,|d\Phi| \geq 2\hat{Q} > 0$ at each point in E.

By enlarging the domain E slightly, we obtain an open, connected domain $\hat{E}$ with smooth boundary such that the closure of E is contained in $\hat{E}$ and $\hat{E} \setminus E_0$ is a bounded subset of (M, g). **From now on, we work exclusively on the closure of $\hat{E}$.**

Step 2: Solving a linear PDE on the enlarged domain $\hat{E}$

By definition of an n-dataset, we have

$$\int_{\hat{E}} \rho\,|df|^2 + \frac{1}{2}\int_{\hat{E}} \rho\,\Big(R - 2\,\Delta\log\rho - \frac{n+1}{n+2}\,|d\log\rho|^2\Big)\,f^2 \geq \int_{\hat{E}} \rho\,Q\,f^2$$

for every smooth test function f on $\hat{E}$ with the property that $f = 0$ on $\partial\hat{E}$ and f is constant near infinity.

By adapting ideas from Eichmair-Huang-Lee-Schoen, we can construct a nonnegative function $\hat{v}$ on the closure of $\hat{E}$ such that $\hat{v} = 0$ on $\partial\hat{E}$, and $\hat{v} = 1 + \gamma\,r^{2-n} + O(r^{2-n-2\delta})$ near infinity, and

$$-\Delta\hat{v} - \langle d\log\rho, d\hat{v}\rangle + \frac{1}{2}\Big(R - 2\,\Delta\log\rho - \frac{n+1}{n+2}\,|d\log\rho|^2 - Q\Big)\,\hat{v} = 0$$

on $\hat{E}$. Using $\hat{v}$ as a test function, we obtain $\gamma < 0$. By the strict maximum principle, $\hat{v} > 0$ in $\hat{E}$.

Step 3: Constructing barriers near infinity

We define a positive function $\hat{\rho}$ on $\hat{E}$ by

$$\hat{\rho} = \rho\, \hat{v}.$$

Then $\hat{\rho} = 1 + \hat{\beta}\, r^{2-n} + O(r^{2-n})$, where $\hat{\beta} = \beta + \gamma$. Note that $\gamma < 0$. By assumption,

$$(n-1)\alpha + 2\beta \leq 0,$$

where α is the coefficient in the asymptotic expansion of the metric and β is the coefficient in the asymptotic expansion of ρ. Thus,

$$(n-1)\alpha + 2\hat{\beta} < 0.$$

Using this inequality, we can construct barriers. Specifically, for λ sufficiently large, we can construct a hypersurface $N_\lambda^+ \approx \{x_n = \lambda\}$ and a hypersurface $N_\lambda^- \approx \{x_n = -\lambda\}$ such that

$$H_{N_\lambda^+} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^+} \rangle > 0$$

and

$$H_{N_\lambda^-} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^-} \rangle > 0.$$

Step 4: Constructing a μ-bubble in E

We now fix $\lambda > 0$ sufficiently large. We consider a slab $E_{\mathrm{slab},\lambda} \subset E$ with boundary $\partial E_{\mathrm{slab},\lambda} = N_\lambda^+ \cup N_\lambda^- \cup \partial E$.

Let Φ be defined as in the Lesourd-Unger-Yau lemma. Recall that $\Phi \to -\infty$ on ∂E. Moreover, N_λ^+ and N_λ^- satisfy

$$H_{N_\lambda^+} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^+} \rangle > 0$$

and

$$H_{N_\lambda^-} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^-} \rangle > 0.$$

Using these barriers, we can construct a μ-bubble in the slab $E_{\mathrm{slab},\lambda}$. We denote by $\mathcal{S}$ the singular set of the μ-bubble, and by Σ the regular set of the μ-bubble. Then Σ is a smooth (possibly incomplete) hypersurface satisfying

$$H_\Sigma + \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle = \Phi.$$

To construct the μ-bubble, we use ideas of Eichmair-Körber; we solve a sequence of free boundary problems and pass to the limit.

Step 5: The stability inequality on Σ

Since Σ arises as a limit of minimizers for the free boundary problem, we obtain a stability inequality which holds away from the singular set.

Proposition: Suppose that a is a real number and f is a smooth test function on Σ with the property that f vanishes near the singular set and $f = a \, \langle \frac{\partial}{\partial x_n}, \nu_\Sigma \rangle$ near infinity. Then

$$\begin{aligned} &\int_\Sigma \hat{\rho} \, |\nabla^\Sigma f|^2 - \int_\Sigma \hat{\rho} \, \mathrm{Ric}(\nu_\Sigma, \nu_\Sigma) \, f^2 - \int_\Sigma \hat{\rho} \, |h_\Sigma|^2 \, f^2 \\ &+ \int_\Sigma \hat{\rho} \, (D^2 \log \hat{\rho})(\nu_\Sigma, \nu_\Sigma) \, f^2 - \int_\Sigma \hat{\rho} \, \langle \nabla \Phi, \nu_\Sigma \rangle \, f^2 \geq 0. \end{aligned}$$

A crucial point is that we allow test functions that are asymptotically constant near infinity.

Step 6: A generalized Schoen-Yau identity on Σ

A lengthy calculation (which uses the Gauss equations and the PDE for $\hat{v}$) gives

$$\begin{aligned}
&\mathrm{Ric}(\nu_\Sigma, \nu_\Sigma) + |h_\Sigma|^2 - (D^2 \log \hat{\rho})(\nu_\Sigma, \nu_\Sigma) + \langle \nabla \Phi, \nu_\Sigma \rangle \\
&+ \frac{1}{2} \Big(R_\Sigma - 2\, \Delta_\Sigma \log \hat{\rho} - \frac{n}{n+1} \, |\nabla^\Sigma \log \hat{\rho}|^2 \Big) \\
&= \frac{1}{2}\, Q + \frac{1}{4}\, \Phi^2 + \langle \nabla \Phi, \nu_\Sigma \rangle + \frac{1}{2} \Big(|h_\Sigma|^2 - \frac{1}{n-1}\, H_\Sigma^2 \Big) \\
&+ \frac{n+2}{2(n+1)} \Big| \nabla \log \hat{v} + \frac{1}{n+2}\, \nabla \log \rho \Big|^2 \\
&+ \frac{n+1}{4(n-1)} \Big(H_\Sigma + \frac{n-1}{n+1} \, \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle \Big)^2.
\end{aligned}$$

The expression on the right hand side is bounded from below by $\frac{1}{2}\, Q + \frac{1}{4}\, \Phi^2 - |\nabla \Phi|$. It follows from our choice of Φ that

$$\frac{1}{2}\, Q + \frac{1}{4}\, \Phi^2 - |\nabla \Phi| \geq \hat{Q} > 0.$$

Step 7: Positivity of a quadratic form on Σ

Combining the stability inequality for Σ and the generalized Schoen-Yau identity, we can draw the following conclusion:

Proposition: Let f be a smooth test function on Σ with the property that f vanishes near the singular set and f is constant near infinity. Then

$$\int_\Sigma \hat{\rho}\,|\nabla^\Sigma f|^2 + \frac{1}{2}\int_\Sigma \hat{\rho}\left(R_\Sigma - 2\,\Delta_\Sigma \log \hat{\rho} - \frac{n}{n+1}\,|\nabla^\Sigma \log \hat{\rho}|^2\right) f^2$$
$$\geq \int_\Sigma \hat{\rho}\,\hat{Q}\,f^2.$$

Here, $\hat{Q}$ is defined as in the Lesourd-Unger-Yau lemma. In particular, $\hat{Q}$ is a smooth positive function.

As above, it is crucial that we can allow test functions that are asymptotically constant near infinity.

Let $\hat{g}$ denote the induced metric on Σ. The Riemannian manifold $(\Sigma, \hat{g})$ may be incomplete due to the presence of singularities.

Step 8: Constructing a function on ambient space that blows up at the singular set at a controlled rate

Proposition: We can find a nonnegative smooth function $\Psi : E \setminus \mathcal{S}$ with the following properties:

- The function Ψ vanishes near infinity.
- If $d_{(M,g)}(x, \mathcal{S})$ is sufficiently small, then

$$\Psi(x) \geq c\, d_{(M,g)}(x, \mathcal{S})^{-2}.$$

- In a neighborhood of the singular set, the function Ψ is a supersolution of a certain linear PDE on Σ. More precisely, if $x \in \Sigma$ and $d_{(M,g)}(x, \mathcal{S})$ is sufficiently small, then

$$\Delta_\Sigma \Psi + \frac{n-3}{n+1} \langle \nabla^\Sigma \log \hat{\rho}, \nabla^\Sigma \Psi \rangle < 0$$

  at the point x.

The function Ψ is constructed in terms of ambient distance functions. Here, we use in a crucial way the work of Cheeger and Naber on the Minkowski dimension of the singular set $\mathcal{S}$.

Step 9: The conformal blow-up technique and the completion the inductive step

Let Ψ be defined as above. We define

$$w = 1 + \varepsilon_0 \, \Psi|_\Sigma,$$

where $\varepsilon_0 > 0$ is chosen small enough so that

$$\Delta_\Sigma \Psi + \frac{n-3}{n+1} \, \langle \nabla^\Sigma \log \hat{\rho}, \nabla^\Sigma \Psi \rangle \leq \frac{n-3}{n+1} \, \varepsilon_0^{-1} \, \hat{Q}$$

at each point on Σ. Such a choice of ε_0 is possible since the expression on the left hand side is nonpositive near the singular set and near infinity.

We define a conformal metric $\tilde{g}$ on Σ by

$$\tilde{g} = w^{\frac{n+1}{n-3}} \, \hat{g}.$$

Since $\Psi(x) \geq c \, d_{(M,g)}(x, \mathcal{S})^{-2}$ near the singular set, the Riemannian manifold $(\Sigma, \tilde{g})$ is complete. Moreover, we define

$$\tilde{\rho} = w^{-\frac{n+1}{2}} \, \hat{\rho}, \qquad \tilde{Q} = \frac{1}{2} \, w^{-\frac{n+1}{n-3}} \, \hat{Q}.$$

Step 9: The conformal blow-up technique and the completion the inductive step (continued)

It follows from our choice of Ψ and ε_0 that $w \geq 1$ and

$$\Delta_\Sigma w + \frac{n-3}{n+1} \, \langle \nabla^\Sigma \log \hat{\rho}, \nabla^\Sigma w \rangle \leq \frac{n-3}{n+1} \, \hat{Q} \leq \frac{n-3}{n+1} \, \hat{Q} \, w.$$

Using this inequality, we can show:

Proposition: Let f be a smooth test function on Σ with the property that f vanishes near the singular set and f is constant near infinity. Then

$$\begin{aligned}
&\int_\Sigma \tilde{\rho} \, |df|_{\tilde{g}}^2 \, d\mathrm{vol}_{\tilde{g}} \\
&+ \frac{1}{2} \int_\Sigma \tilde{\rho} \left(R_{\tilde{g}} - 2 \, \Delta_{\tilde{g}} \log \tilde{\rho} - \frac{n}{n+1} \, |d \log \tilde{\rho}|_{\tilde{g}}^2 \right) f^2 \, d\mathrm{vol}_{\tilde{g}} \\
&\geq \int_\Sigma \tilde{\rho} \, \tilde{Q} \, f^2 \, d\mathrm{vol}_{\tilde{g}}.
\end{aligned}$$

Step 9: The conformal blow-up technique and the completion the inductive step (continued)

To summarize, $(\Sigma, \tilde{g}, \tilde{\rho}, \tilde{Q})$ is an $(n-1)$-dataset. The metric $\tilde{g}$ satisfies

$$\tilde{g}_{ij} = \delta_{ij} + O(r^{2-n})$$

near infinity. Moreover,

$$\tilde{\rho} = 1 + O(r^{2-n})$$

near infinity. Consequently, the mass of the $(n-1)$-dataset $(\Sigma, \tilde{g}, \tilde{\rho}, \tilde{Q})$ is equal to 0. This contradicts the inductive hypothesis, thereby completing the inductive step.

The spacetime positive energy theorem

Theorem (B.-Wang April 2026): Let (M, g) be an asymptotically flat manifold of dimension $n \geq 4$, so that

$$g = (1 + \alpha\, r^{2-n})\, \bar{g} + O(r^{2-n-2\delta})$$

near infinity. Suppose that q is a symmetric $(0, 2)$-tensor on M satisfying $\mu - |J| > 0$, where $\mu = \frac{1}{2}\,(R - |q|^2 + \mathrm{tr}(q)^2)$ and $J = \mathrm{div}\, q - d\, \mathrm{tr}(q)$. Then $\alpha > 0$.

For $n = 3$, the spacetime positive energy theorem was famously proved by Schoen-Yau using the Jang equation. In an remarkable work, Eichmair extended the proof to dimension $n \leq 7$, again using the Jang equation. Eichmair-Huang-Lee-Schoen gave an alternative proof for $n \leq 7$ using marginally outer trapped surfaces.

We work with a modified Jang equation that includes a capillary term. This PDE is similar to the one considered by Schoen-Yau and Eichmair.

The spacetime positive mass theorem and the rigidity statement

In view of the density theorem of Eichmair-Huang-Lee-Schoen (JEMS 2015), our theorem implies the usual spacetime positive energy theorem for initial data sets satisfying the dominant energy condition $\mu - |J| \geq 0$. Moreover, the density theorem of Eichmair-Huang-Lee-Schoen makes it possible to weaken the asymptotic assumptions.

Furthermore, as explained in the final remark on p. 119 of Eichmair-Huang-Lee-Schoen's paper, one may use the boost argument of Christodoulou and O'Murchadha to deduce the spacetime positive mass theorem from the spacetime positive energy theorem. In light of that, our result implies the spacetime positive mass theorem in dimension $n \geq 4$.

Finally, in the case of equality, (M, g, q) arises as a spacelike hypersurface in the Minkowski spacetime. This follows directly from a theorem of Huang and Lee.

The hyperbolic positive energy theorem

The positive energy theorem for asymptotically hyperbolic manifolds follows from the spacetime positive energy theorem for asymptotically flat initial data sets satisfying the dominant energy condition. This was shown in a remarkable work of Chruściel and Delay in 2019. In combination with the work of Chruściel-Delay, our result implies the hyperbolic positive energy theorem in dimension $n \geq 4$.