

A DIMENSION DESCENT SCHEME FOR THE POSITIVE MASS THEOREM IN ARBITRARY DIMENSION

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ABSTRACT. We describe how the Schoen-Yau proof of the positive mass theorem can be extended to arbitrary dimensions. To overcome the problem of singularities, we propose a new inductive scheme. To carry out the inductive step, we use a combination of several techniques, including the shielding principle of Lesourd-Unger-Yau, as well as a conformal blow-up argument in the spirit of Bi-Hao-He-Shi-Zhu. Our arguments also rely on the Cheeger-Naber bound for the Minkowski dimension of the singular set.

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1. INTRODUCTION

We begin with several definitions.

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Definition 1.1. Let $n \geq 3$ be an integer. Let (M, g) be a complete Riemannian manifold of dimension n . We say that (M, g) has an asymptotically flat end if there exists a compact domain $K \subset M$ with smooth boundary and a connected component E_0 of $M \setminus K$ such that E_0 is diffeomorphic to the complement of the unit ball in \mathbb{R}^n . Moreover, we require that there exist real numbers α and $\delta > 0$ such that

$$|\bar{D}^m(g - (1 + \alpha r^{2-n})\bar{g})|_{\bar{g}} \leq C(m) r^{2-n-m-2\delta}$$

at each point in E_0 and for every nonnegative integer m . Here, \bar{g} denotes the Euclidean metric on the asymptotically flat end E_0 , \bar{D}^m denotes the covariant derivative of order m with respect to \bar{g} , and $r = \sqrt{x_1^2 + \dots + x_n^2}$ denotes the radial coordinate on the asymptotically flat end E_0 .

Definition 1.2. Let $n \geq 3$ be an integer. Let (M, g) be a complete Riemannian manifold of dimension n with an asymptotically flat end. We define the mass of (M, g) to be $(n-1)\alpha$, where α denotes the coefficient in the asymptotic expansion of the metric.

Definition 1.3. Let $n \geq 3$ be an integer. An n -dataset consists of a complete Riemannian manifold (M, g) of dimension n together with positive smooth functions ρ and Q satisfying the following conditions:

- The manifold (M, g) has an asymptotically flat end E_0 .
- If $n = 3$, we assume in addition that $M \setminus E_0$ is a bounded subset of (M, g) . In other words, if $n = 3$, we assume that (M, g) has no ends other than E_0 .
- There exist real numbers β and $\delta > 0$ such that

$$|\bar{D}^m(\rho - (1 + \beta r^{2-n}))|_{\bar{g}} \leq C(m) r^{2-n-m-2\delta}$$

and

$$|\bar{D}^m Q|_{\bar{g}} \leq C(m) r^{-n-m-2\delta}$$

at each point in E_0 and for every nonnegative integer m .

- We have

$$\begin{aligned} & \int_M \rho |df|^2 + \frac{1}{2} \int_M \rho \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 \right) f^2 \\ & \geq \int_M \rho Q f^2 \end{aligned}$$

for every smooth test function f with the property that the set $\{f \neq 0\} \setminus E_0$ is bounded and there exists a constant a such that the set $\{f \neq a\} \cap E_0$ is bounded. In view of our decay conditions, the functions R , $\Delta \log \rho$, $|d \log \rho|^2$, and Q belong to $L^1(E_0)$, so the integrals are well-defined.

Definition 1.4. Let $n \geq 3$ be an integer, and let (M, g, ρ, Q) be an n -dataset. We define the mass of the n -dataset (M, g, ρ, Q) to be $(n-1)\alpha + 2\beta$, where α denotes the coefficient in the asymptotic expansion of the metric and β denotes the coefficient in the asymptotic expansion of ρ .

We now state the main result of this paper.

Theorem 1.5. *Let $n \geq 3$ be an integer, and let (M, g, ρ, Q) be an n -dataset. Then the mass (in the sense of Definition 1.4) of the n -dataset (M, g, ρ, Q) is strictly positive.*

As a special case, we obtain the following result.

Corollary 1.6. *Let $n \geq 3$ be an integer. Suppose that (M, g) is a complete Riemannian manifold with an asymptotically flat end. If the scalar curvature of g is positive at each point in M , then the mass (in the sense of Definition 1.2) of (M, g) is strictly positive.*

Corollary 1.6 follows from Theorem 1.5 by putting $\rho = 1$ and $Q = \frac{1}{2} R$.

In their groundbreaking works [18],[19],[20], Schoen and Yau proved the positive mass theorem for asymptotically flat manifolds of dimension $n \leq 7$. Lesourd, Unger, and Yau [14] extended the positive mass theorem to manifolds which have one asymptotically flat end and in addition have other arbitrary ends. As part of their work, Lesourd, Unger, and Yau introduced a shielding principle, which plays a central role in our work. Chodosh, Mantoulidis, Schulze, and Wang [8] have verified the positive mass theorem up to dimension 11, and Bi, Hao, He, Shi, and Zhu [4] recently gave a proof of the positive mass theorem up to dimension 19. Finally, Schoen and Yau [21] and Lohkamp [15] have proposed proofs of the positive mass theorem in arbitrary dimension.

The proof of Theorem 1.5 proceeds by induction on n . For $n = 3$, Theorem 1.5 can be reduced to the classical positive mass theorem of Schoen and Yau [19].

We now give an overview of the proof of the inductive step. Suppose that $n \geq 4$ and (M, g, ρ, Q) is an n -dataset with nonpositive mass. Following Lesourd-Unger-Yau, we construct an open domain E together with smooth functions Φ and \hat{Q} satisfying the following conditions:

- The closure of E_0 is contained in E .
- The complement $E \setminus E_0$ is a bounded subset of (M, g) .
- $\Phi = 0$ and $\hat{Q} = \frac{1}{2} Q$ at each point in E_0 .
- $\Phi \leq 0$ and $\hat{Q} > 0$ at each point in E .
- $\Phi \rightarrow -\infty$ on the boundary ∂E .
- $Q + \frac{1}{2} \Phi^2 - 2|d\Phi| \geq 2\hat{Q}$ at each point in E .

Note that E has compact, smooth boundary and one asymptotically flat end.

In the next step, we slightly enlarge the domain E . On the enlarged domain \hat{E} , we construct a positive solution \hat{v} of the linear PDE

$$-\Delta \hat{v} - \langle d \log \rho, d \hat{v} \rangle + \frac{1}{2} \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) \hat{v} = 0$$

with Dirichlet boundary condition on $\partial \hat{E}$. We then restrict \hat{v} to the smaller domain E , and define $\hat{\rho} = \rho \hat{v}$.

In the next step, we construct a μ -bubble in E . This μ -bubble may have singularities. We denote by Σ the regular part of the μ -bubble. Then Σ is a smooth hypersurface in E satisfying

$$H_\Sigma + \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle = \Phi.$$

To construct this μ -bubble we need barriers near infinity, as well as barriers near the boundary ∂E . To construct the barriers near infinity, we use the fact that the n -dataset (M, g, ρ, Q) has nonpositive mass. To construct the barriers near ∂E , we use the fact that $\Phi \rightarrow -\infty$ on ∂E .

The hypersurface Σ with its induced metric \hat{g} can be viewed as an incomplete manifold of dimension $n-1$ with an asymptotically flat end. Moreover, Σ satisfies a stability inequality. By combining the stability inequality for Σ with a generalization of the famous Schoen-Yau identity, we conclude that a certain quadratic form on Σ is positive.

In the last step, we construct a conformal metric $\tilde{g} = w^{\frac{n+1}{n-3}} \hat{g}$ on Σ . The conformal factor w is obtained by restricting a suitable function on ambient space to Σ . This function blows up at a controlled rate near the singular set, thereby ensuring that the metric \tilde{g} is complete. Importantly, the conformal factor can be chosen in such a way that the positivity of the quadratic form is preserved. This allows us to construct an $(n-1)$ -dataset with mass equal to 0, thereby completing the inductive step.

2. PROOF OF THEOREM 1.5 FOR $n = 3$

Throughout this section, we assume that (M, g, ρ, Q) is a 3-dataset. Let E_0 denote the asymptotically flat end of (M, g) . By assumption, $M \setminus E_0$ is a bounded subset of (M, g) .

Proposition 2.1. *Suppose that a is a constant and F is a smooth function on M such that $F = a\rho^{\frac{1}{2}}$ near infinity. Then*

$$\int_M |dF|^2 + \frac{1}{8} \int_M R F^2 - \frac{1}{4} \int_M Q F^2 \geq -\pi\beta a^2,$$

where β denotes the coefficient in the asymptotic expansion of ρ .

Proof. Let $f = \rho^{-\frac{1}{2}} F$. Then $f = a$ near infinity. Since (M, g, ρ, Q) is a 3-dataset, we know that

$$(1) \quad \int_M \rho |df|^2 + \frac{1}{2} \int_M \rho \left(R - 2 \Delta \log \rho - \frac{4}{5} |d \log \rho|^2 \right) f^2 \geq \int_M \rho Q f^2.$$

On the other hand, it follows from the divergence theorem that

$$(2) \quad \int_M \operatorname{div}(f^2 d\rho) = -4\pi\beta a^2.$$

Adding (1) and (2) gives

$$\begin{aligned} & \int_M \rho |df|^2 + 2 \int_M f \langle d\rho, df \rangle + \frac{3}{5} \int_M \rho^{-1} |d\rho|^2 f^2 + \frac{1}{2} \int_M \rho R f^2 \\ & \geq -4\pi\beta a^2 + \int_M \rho Q f^2. \end{aligned}$$

Using the pointwise inequality

$$\begin{aligned} & \rho |df|^2 + 2f \langle d\rho, df \rangle + \frac{3}{5} \rho^{-1} |d\rho|^2 f^2 \\ & = \frac{8}{3} \rho \left| df + \frac{1}{2} \rho^{-1} f d\rho \right|^2 - \frac{5}{3} \rho \left| df + \frac{1}{5} \rho^{-1} f d\rho \right|^2 \\ & \leq 4 \rho \left| df + \frac{1}{2} \rho^{-1} f d\rho \right|^2, \end{aligned}$$

we obtain

$$4 \int_M \rho \left| df + \frac{1}{2} \rho^{-1} f d\rho \right|^2 + \frac{1}{2} \int_M \rho R f^2 \geq -4\pi\beta a^2 + \int_M \rho Q f^2.$$

Since $F = \rho^{\frac{1}{2}} f$, we conclude that

$$4 \int_M |dF|^2 + \frac{1}{2} \int_M R F^2 \geq -4\pi\beta a^2 + \int_M Q F^2.$$

This completes the proof of Proposition 2.1.

In the next step, we construct a solution of a certain linear PDE. To that end, we follow the arguments in Eichmair-Huang-Lee-Schoen [9] and Carlotto [6]. Let us fix a nonnegative smooth function ω such that ω is supported in E_0 and $\omega = r^{-2}$ near infinity. By Hardy's inequality (see e.g. [24], Theorem 6.4.10), we can find a positive constant κ such that

$$(3) \quad \int_M |dF|^2 + \frac{1}{8} \int_M Q F^2 \geq \kappa \int_M \omega F^2$$

for every smooth function F on M that vanishes near infinity. Since the function $4\kappa\omega + R$ is positive near infinity, we can find a large constant $\Lambda > 2$ such that

$$(4) \quad (2\Lambda - 4)Q + 4\kappa\omega + R \geq 0$$

at each point on M .

Proposition 2.2 (Coercivity). *Suppose that F is a smooth function on M that vanishes near infinity. Then*

$$\int_M |dF|^2 + \frac{1}{8} \int_M R F^2 \geq \frac{1}{8\Lambda} \int_M (Q + 4\kappa\omega) F^2.$$

Proof. We compute

$$\begin{aligned}
& \int_M |dF|^2 + \frac{1}{8} \int_M R F^2 - \frac{1}{8\Lambda} \int_M (Q + 4\kappa\omega) F^2 \\
&= \frac{\Lambda - 1}{\Lambda} \left(\int_M |dF|^2 + \frac{1}{8} \int_M R F^2 - \frac{1}{4} \int_M Q F^2 \right) \\
&+ \frac{1}{\Lambda} \left(\int_M |dF|^2 + \frac{1}{8} \int_M Q F^2 - \kappa \int_M \omega F^2 \right) \\
&+ \frac{1}{8\Lambda} \int_M \left((2\Lambda - 4) Q + 4\kappa\omega + R \right) F^2.
\end{aligned}$$

The first term on the right hand side is nonnegative by Proposition 2.1. The second term on the right hand side is nonnegative by the Hardy inequality (3). The third term on the right hand side is nonnegative in view of (4). This completes the proof of Proposition 2.2.

Let \mathcal{H} denote the set of all functions $F \in H_{\text{loc}}^1(M)$ with the property that

$$\int_M |dF|^2 + \int_M (Q + \omega) F^2 < \infty.$$

We define

$$\|F\|_{\mathcal{H}}^2 = \int_M |dF|^2 + \int_M (Q + \omega) F^2$$

for all $F \in \mathcal{H}$. Since $|R| \leq O(r^{-3-2\delta})$, we know that $R \in L^1(M)$. Moreover, $R F \in L^1(M)$ and $R F^2 \in L^1(M)$ for all $F \in \mathcal{H}$.

Proposition 2.3. *We can find a function $v \in \mathcal{H}$ with the property that v minimizes the functional*

$$\int_M |dv|^2 + \frac{1}{8} \int_M R v^2 + \frac{1}{4} \int_M R v$$

among all functions $v \in \mathcal{H}$. Moreover, we can choose v so that $1 + v \geq 0$.

Proof. It follows from Proposition 2.2 and a standard approximation argument that

$$\int_M |dF|^2 + \frac{1}{8} \int_M R F^2 \geq \frac{1}{8\Lambda} \int_M (Q + 4\kappa\omega) F^2$$

for all $F \in \mathcal{H}$. Using this coercivity property, the existence of a minimizer follows easily. By replacing v by $|1 + v| - 1$, we can arrange that $1 + v$ is nonnegative. This completes the proof of Proposition 2.3.

Let v denote the minimizer constructed in Proposition 2.3. By elliptic regularity theory, v is a smooth solution of the PDE

$$(5) \quad -\Delta v + \frac{1}{8} R(1 + v) = 0.$$

Since $v \in \mathcal{H}$, the function $1 + v$ does not vanish identically. Moreover, the function $1 + v$ is nonnegative. Using the strict maximum principle, we conclude that the function $1 + v$ is strictly positive everywhere.

Our assumptions imply that

$$|\bar{D}^m R|_{\bar{g}} \leq O(r^{-3-m-2\delta})$$

for every nonnegative integer m . Standard results for linear PDE [17] imply that there exists a real number γ such that

$$|\bar{D}^m(v - \gamma r^{-1})|_{\bar{g}} \leq O(r^{-1-m-2\hat{\delta}})$$

for every nonnegative integer m , where $\hat{\delta} \in (0, \delta)$.

Proposition 2.4. *We have $4\gamma < \beta$, where β denotes the coefficient in the asymptotic expansion of ρ and γ denotes the coefficient in the asymptotic expansion of v .*

Proof. Proposition 2.1 implies that

$$\int_M |dF|^2 + \frac{1}{8} \int_M R F^2 - \frac{1}{4} \int_M Q F^2 \geq -\pi\beta$$

for every smooth function F on M with the property that $F = \rho^{\frac{1}{2}}$ near infinity. Note that $\rho = 1 + O(r^{-1})$ and $1 + v = 1 + O(r^{-1})$, and we have corresponding estimates for all the higher derivatives. By a standard approximation argument, the preceding inequality holds for the function $F = 1 + v$. This gives

$$\int_M |dv|^2 + \frac{1}{8} \int_M R(1+v)^2 - \frac{1}{4} \int_M Q(1+v)^2 \geq -\pi\beta.$$

On the other hand, using (5), we obtain

$$\int_M |dv|^2 + \frac{1}{8} \int_M R(1+v)^2 = \int_M \operatorname{div}((1+v)dv) = -4\pi\gamma.$$

Finally, since the function Q is strictly positive and the function $1 + v$ is strictly positive, we know that

$$\int_M Q(1+v)^2 > 0.$$

Putting these facts together, we conclude that $-4\pi\gamma > -\pi\beta$. This completes the proof of Proposition 2.4.

It follows from (5) that the conformal metric $(1+v)^4 g$ has zero scalar curvature. Moreover,

$$(1+v)^4 g = (1 + (\alpha + 4\gamma)r^{-1})\bar{g} + O(r^{-1-\hat{\delta}})$$

near infinity. We now apply the classical positive mass theorem of Schoen and Yau [19] to the conformal metric $(1+v)^4 g$. This implies $\alpha + 4\gamma \geq 0$. Combining this inequality with Proposition 2.4, we conclude that $\alpha + \beta > 0$. Therefore, the 3-dataset (M, g, ρ, Q) has strictly positive mass in the sense

of Definition 1.4. This completes the proof of Theorem 1.5 in the special case $n = 3$.

3. PROOF OF THEOREM 1.5 FOR $n \geq 4$

In this section, we complete the proof of Theorem 1.5. Let us fix an integer $n \geq 4$. We assume that Theorem 1.5 holds for all $(n - 1)$ -datasets. We will show that Theorem 1.5 holds for all n -datasets. We argue by contradiction. Suppose that (M, g, ρ, Q) is an n -dataset with nonpositive mass. In other words,

$$(6) \quad (n - 1)\alpha + 2\beta \leq 0,$$

where α denotes the coefficient in the asymptotic expansion of the metric and β denotes the coefficient in the asymptotic expansion of ρ . Let E_0 denote the asymptotically flat end of M .

3.1. A result from Lesourd-Unger-Yau's work. In this subsection, we recall an important construction from the work of Lesourd-Unger-Yau.

Lemma 3.1 (cf. Lesourd-Unger-Yau [14], Proposition 3.1). *We can find an open, connected domain E with smooth boundary, a smooth function Φ defined on E , and a smooth function \hat{Q} defined on E with the following properties:*

- The closure of E_0 is contained in E .
- The complement $E \setminus E_0$ is a bounded subset of (M, g) .
- $\Phi = 0$ and $\hat{Q} = \frac{1}{2}Q$ at each point in E_0 .
- $\Phi \leq 0$ and $\hat{Q} > 0$ at each point in E .
- $\Phi \rightarrow -\infty$ on the boundary ∂E .
- $Q + \frac{1}{2}\Phi^2 - 2|d\Phi| \geq 2\hat{Q}$ at each point in E .

Proof. Let us fix positive real numbers s_0 and s_1 such that $s_1 \geq s_0$ and

$$Q(x) > \frac{128}{s_1 s_0}$$

for each point $x \in \mathcal{N}_{(M, g)}(E_0, 2s_0) \setminus E_0$. Let us fix a smooth function $\chi : \mathbb{R} \rightarrow [0, 1]$ such that $\chi = 0$ on $(-\infty, \frac{1}{2}]$, $\chi = 1$ on $[1, \infty)$, and $0 \leq \chi' \leq 3$. We define a smooth function $\varphi : (-\infty, s_1 + s_0) \rightarrow (-\infty, 0]$ by

$$\varphi(s) = -\frac{8}{s_1 + s_0 - s} \chi\left(\frac{s}{s_0}\right)$$

for all $s \in (-\infty, s_1 + s_0)$. Clearly, φ is monotone decreasing, $\varphi(s) = 0$ for all $s \in (-\infty, \frac{s_0}{2}]$, and $\varphi(s) = -\frac{8}{s_1 + s_0 - s}$ for all $s \in [s_0, s_1 + s_0)$. In particular,

$\varphi(s) \rightarrow -\infty$ as $s \rightarrow s_1 + s_0$. Moreover,

$$\begin{aligned}\varphi'(s) &= -\frac{8}{(s_1 + s_0 - s)s_0} \chi'\left(\frac{s}{s_0}\right) - \frac{8}{(s_1 + s_0 - s)^2} \chi\left(\frac{s}{s_0}\right) \\ &\geq -\frac{24}{s_1 s_0} - \frac{8}{s_1^2} \\ &\geq -\frac{32}{s_1 s_0}\end{aligned}$$

for all $s \in (-\infty, s_0)$ and

$$\varphi'(s) = -\frac{1}{8} \varphi(s)^2$$

for all $s \in [s_0, s_1 + s_0)$.

By Proposition 2.1 in [12], we can find a smooth function σ on M such that $|\sigma(x) - d_{(M,g)}(x, E_0)| < \frac{s_0}{2}$ and $|d\sigma(x)| < 2$ for all $x \in M$. We may further assume that $s_1 + s_0$ is a regular value of σ . We claim that

$$(7) \quad Q(x) + \frac{1}{2} \varphi(\sigma(x))^2 + 4\varphi'(\sigma(x)) > 0$$

for each point $x \in \{\sigma < s_1 + s_0\}$. To prove this, we distinguish three cases:

Case 1: Suppose that $x \in E_0$. In this case, $\sigma(x) < \frac{s_0}{2}$. This implies

$$Q(x) + \frac{1}{2} \varphi(\sigma(x))^2 + 4\varphi'(\sigma(x)) = Q(x) > 0.$$

Case 2: Suppose that $x \in \{\sigma < s_0\} \setminus E_0$. Then $x \in \mathcal{N}_{(M,g)}(E_0, 2s_0) \setminus E_0$. Consequently, $Q(x) > \frac{128}{s_1 s_0}$. This gives

$$Q(x) + \frac{1}{2} \varphi(\sigma(x))^2 + 4\varphi'(\sigma(x)) > \frac{128}{s_1 s_0} + 4\varphi'(\sigma(x)) \geq 0.$$

Case 3: Suppose that $x \in \{s_0 \leq \sigma < s_1 + s_0\}$. In this case,

$$Q(x) + \frac{1}{2} \varphi(\sigma(x))^2 + 4\varphi'(\sigma(x)) = Q(x) > 0.$$

This proves (7).

Let E denote the connected component of the set $\{\sigma < s_1 + s_0\}$ that contains the set E_0 . Then $E \subset \mathcal{N}_{(M,g)}(E_0, s_1 + 2s_0)$. In particular, the set $E \setminus E_0$ is a bounded subset of (M, g) . We define a smooth function Φ on E by

$$\Phi(x) = \varphi(\sigma(x))$$

for $x \in E$. Clearly, $\Phi = 0$ at each point in E_0 , $\Phi \leq 0$ at each point in E , and $\Phi \rightarrow -\infty$ on the boundary ∂E . We define a smooth function \hat{Q} on E by

$$\hat{Q}(x) = \frac{1}{2} Q(x) + \frac{1}{4} \varphi(\sigma(x))^2 + 2\varphi'(\sigma(x))$$

for $x \in E$. Clearly, $\hat{Q} = \frac{1}{2}Q$ at each point in E_0 . Moreover, it follows from (7) that $\hat{Q} > 0$ at each point in E . Finally,

$$\begin{aligned} & Q(x) + \frac{1}{2}\Phi(x)^2 - 2|d\Phi(x)| - 2\hat{Q}(x) \\ &= Q(x) + \frac{1}{2}\varphi(\sigma(x))^2 - 2|\varphi'(\sigma(x))||d\sigma(x)| - 2\hat{Q}(x) \\ &= 2|\varphi'(\sigma(x))|(2 - |d\sigma(x)|) \\ &\geq 0 \end{aligned}$$

for each point $x \in E$. This completes the proof of Lemma 3.1.

Remark 3.2. In Lemma 3.1, we allow the possibility that $\partial E = \emptyset$.

In the following, we fix an open, connected domain \hat{E} with smooth boundary such that the closure of E is contained in \hat{E} and the complement $\hat{E} \setminus E$ is a bounded subset of (M, g) . **From now on, we will work exclusively on the closure of the domain \hat{E} .**

3.2. Solving a linear PDE with Dirichlet boundary condition on the enlarged domain \hat{E} . In this subsection, we construct a solution of a certain linear PDE on \hat{E} with Dirichlet boundary condition. We again follow the arguments in Eichmair-Huang-Lee-Schoen [9] and Carlotto [6]. Let us fix a nonnegative smooth function ω such that ω is supported in E_0 and $\omega = r^{-2}$ near infinity. By Hardy's inequality (see e.g. [24], Theorem 6.4.10), we can find a positive constant κ such that

$$(8) \quad \int_{\hat{E}} \rho |df|^2 + \frac{1}{2} \int_{\hat{E}} \rho Q f^2 \geq \kappa \int_{\hat{E}} \rho \omega f^2$$

for every smooth function f on \hat{E} that vanishes near infinity. Since the function

$$\kappa\omega + R - 2\Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2$$

is positive near infinity, we can find a large constant $\Lambda > 4$ such that

$$(9) \quad (\Lambda - 4)Q + \kappa\omega + R - 2\Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 \geq 0$$

at each point on \hat{E} .

Proposition 3.3 (Coercivity). *Suppose that f is a smooth function on \hat{E} such that $f = 0$ on $\partial\hat{E}$ and f vanishes near infinity. Then*

$$\begin{aligned} & \int_{\hat{E}} \rho |df|^2 + \frac{1}{2} \int_{\hat{E}} \rho \left(R - 2\Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) f^2 \\ & \geq \frac{1}{2\Lambda} \int_{\hat{E}} \rho (Q + \kappa\omega) f^2. \end{aligned}$$

Proof. We compute

$$\begin{aligned}
 & \int_{\hat{E}} \rho |df|^2 + \frac{1}{2} \int_{\hat{E}} \rho \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) f^2 \\
 & - \frac{1}{2\Lambda} \int_{\hat{E}} \rho (Q + \kappa \omega) f^2 \\
 & = \frac{\Lambda - 1}{\Lambda} \left(\int_{\hat{E}} \rho |df|^2 + \frac{1}{2} \int_{\hat{E}} \rho \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - 2Q \right) f^2 \right) \\
 & + \frac{1}{\Lambda} \left(\int_{\hat{E}} \rho |df|^2 + \frac{1}{2} \int_{\hat{E}} \rho Q f^2 - \kappa \int_{\hat{E}} \rho \omega f^2 \right) \\
 & + \frac{1}{2\Lambda} \int_{\hat{E}} \rho \left((\Lambda - 4) Q + \kappa \omega + R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 \right) f^2.
 \end{aligned}$$

The first term on the right hand side is nonnegative since (M, g, ρ, Q) is an n -dataset. The second term on the right hand side is nonnegative by the Hardy inequality (8). The third term on the right hand side is nonnegative in view of (9). This completes the proof of Proposition 3.3.

Let \mathcal{H} denote the set of all functions $f \in H_{\text{loc}}^1(\hat{E})$ with the property that

$$\int_{\hat{E}} \rho |df|^2 + \int_{\hat{E}} \rho (Q + \omega) f^2 < \infty$$

and the boundary trace of f along $\partial \hat{E}$ vanishes. We define

$$\|f\|_{\mathcal{H}}^2 = \int_{\hat{E}} \rho |df|^2 + \int_{\hat{E}} \rho (Q + \omega) f^2$$

for all $f \in \mathcal{H}$.

Let us fix a nonnegative smooth function v_0 on \hat{E} such that $v_0 = 0$ on $\partial \hat{E}$ and $v_0 = 1$ near infinity.

Proposition 3.4. *We can find a function $v \in \mathcal{H}$ with the property that v minimizes the functional*

$$\begin{aligned}
 & \int_{\hat{E}} \rho |dv|^2 + \frac{1}{2} \int_{\hat{E}} \rho \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) v^2 \\
 & + 2 \int_{\hat{E}} \rho \langle dv_0, dv \rangle + \int_{\hat{E}} \rho \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) v_0 v
 \end{aligned}$$

among all functions $v \in \mathcal{H}$. Moreover, we can choose v so that $v_0 + v \geq 0$.

Proof. It follows from Proposition 3.3 that

$$\begin{aligned}
 & \int_{\hat{E}} \rho |df|^2 + \frac{1}{2} \int_{\hat{E}} \rho \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) f^2 \\
 & \geq \frac{1}{2\Lambda} \int_{\hat{E}} \rho (Q + \kappa \omega) f^2.
 \end{aligned}$$

for all $f \in \mathcal{H}$. Using this coercivity property, the existence of a minimizer follows easily. By replacing v by $|v_0 + v| - v_0$, we can arrange that $v_0 + v \geq 0$

nonnegative. This completes the proof of Proposition 3.4.

Let v denote the minimizer constructed in Proposition 3.4. By elliptic regularity theory, v is a smooth solution of the PDE

$$(10) \quad \begin{aligned} & -\Delta(v_0 + v) - \langle d \log \rho, d(v_0 + v) \rangle \\ & + \frac{1}{2} \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) (v_0 + v) = 0 \end{aligned}$$

on the domain \hat{E} with Dirichlet boundary condition $v = 0$ on $\partial \hat{E}$. Since $v \in \mathcal{H}$, the function $v_0 + v$ does not vanish identically. Moreover, the function $v_0 + v$ is nonnegative. Using the strict maximum principle, we conclude that the function $v_0 + v$ is strictly positive everywhere.

Our assumptions imply that

$$\left| \bar{D}^m \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) \right|_{\bar{g}} \leq O(r^{-n-m-2\delta})$$

for every nonnegative integer m . Standard results for linear PDE [17] imply that there exists a real number γ such that

$$|\bar{D}^m(v - \gamma r^{2-n})|_{\bar{g}} \leq O(r^{2-n-m-2\hat{\delta}})$$

for every nonnegative integer m , where $\hat{\delta} \in (0, \delta)$.

Proposition 3.5. *We have $\gamma < 0$.*

Proof. Since (M, g, ρ, Q) is an n -dataset, we know that

$$\int_{\hat{E}} \rho |df|^2 + \frac{1}{2} \int_{\hat{E}} \rho \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - 2Q \right) f^2 \geq 0$$

for every smooth function f on \hat{E} with the property that $f = 0$ on $\partial \hat{E}$ and $f = 1$ near infinity. By approximation, the preceding inequality also holds for the function $f = v_0 + v$. This gives

$$\int_{\hat{E}} \rho |d(v_0 + v)|^2 + \frac{1}{2} \int_{\hat{E}} \rho \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - 2Q \right) (v_0 + v)^2 \geq 0.$$

On the other hand, using (10), we obtain

$$\begin{aligned} & \int_{\hat{E}} \rho |d(v_0 + v)|^2 + \frac{1}{2} \int_{\hat{E}} \rho \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) (v_0 + v)^2 \\ & = \int_{\hat{E}} \operatorname{div}(\rho (v_0 + v) d(v_0 + v)) = -(n-2) |S^{n-1}| \gamma. \end{aligned}$$

Note that there is no boundary term on $\partial \hat{E}$ since $v_0 + v$ vanishes on $\partial \hat{E}$. Finally, since the function Q is strictly positive and the function $v_0 + v$ is strictly positive, we know that

$$\int_{\hat{E}} \rho Q (v_0 + v)^2 > 0.$$

Putting these facts together, we conclude that $-(n-2)|S^{n-1}|\gamma > 0$. This completes the proof of Proposition 3.5.

From now on, we will work exclusively on the closure of the domain E . We define $\hat{v} = v_0 + v$ and $\hat{\rho} = \rho \hat{v}$. With this understood, \hat{v} and $\hat{\rho}$ are strictly positive smooth functions on the closure of E . The function \hat{v} satisfies

$$|\bar{D}^m(\hat{v} - (1 + \gamma r^{2-n}))|_{\bar{g}} \leq O(r^{2-n-m-2\hat{\delta}})$$

for every nonnegative integer m . Hence, if we put $\hat{\beta} = \beta + \gamma$, then the function $\hat{\rho}$ satisfies

$$|\bar{D}^m(\hat{\rho} - (1 + \hat{\beta} r^{2-n}))|_{\bar{g}} \leq O(r^{2-n-m-2\hat{\delta}})$$

for every nonnegative integer m . Using Proposition 3.5 and the inequality (6), we obtain

$$(11) \quad (n-1)\alpha + 2\hat{\beta} < 0.$$

Finally, the identity (10) gives

$$(12) \quad \begin{aligned} & -\Delta \hat{v} - \langle d \log \rho, d \hat{v} \rangle \\ & + \frac{1}{2} \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) \hat{v} = 0 \end{aligned}$$

on the domain E . This implies

$$(13) \quad \begin{aligned} & -\Delta \log \hat{v} - |d \log \hat{v}|^2 - \langle d \log \rho, d \log \hat{v} \rangle \\ & + \frac{1}{2} \left(R - 2 \Delta \log \rho - \frac{n+1}{n+2} |d \log \rho|^2 - Q \right) = 0 \end{aligned}$$

on the domain E .

3.3. A family of hypersurfaces in the asymptotically flat end E_0 with positive $\hat{\rho}$ -weighted mean curvature. Throughout this subsection, we identify the asymptotically flat end E_0 with the complement of the unit ball in \mathbb{R}^n . For $\lambda > 0$ sufficiently large, we define two hypersurfaces N_λ^+ and N_λ^- by

$$N_\lambda^+ = \left\{ x_n = \lambda - (n-3+\hat{\delta})^{-1} (x_1^2 + \dots + x_{n-1}^2 + \lambda^2)^{-\frac{n-3+\hat{\delta}}{2}} \right\}$$

and

$$N_\lambda^- = \left\{ -x_n = \lambda - (n-3+\hat{\delta})^{-1} (x_1^2 + \dots + x_{n-1}^2 + \lambda^2)^{-\frac{n-3+\hat{\delta}}{2}} \right\}.$$

We choose the unit normal vector field along N_λ^+ so that $dx_n(\nu_{N_\lambda^+}) > 0$ at each point on N_λ^+ . We choose the unit normal vector field along N_λ^- so that $dx_n(\nu_{N_\lambda^-}) < 0$ at each point on N_λ^- . The following proposition is similar to the classical work of Schoen and Yau.

Proposition 3.6. *If $\lambda > 0$ is sufficiently large, then the hypersurface N_λ^+ satisfies*

$$H_{N_\lambda^+} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^+} \rangle > 0$$

and the hypersurface N_λ^- satisfies

$$H_{N_\lambda^-} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^-} \rangle > 0.$$

Here, the mean curvature and unit normal vector are computed with respect to the metric g .

Proof. We only prove the assertion for N_λ^+ . The proof for N_λ^- is analogous. Let $\bar{H}_{N_\lambda^+}$ denote the mean curvature of the hypersurface N_λ^+ with respect to the Euclidean metric \bar{g} . Then

$$\begin{aligned} & \left| \bar{H}_{N_\lambda^+} - (x_1^2 + \dots + x_{n-1}^2 + \lambda^2)^{-\frac{n+1+\hat{\delta}}{2}} (\hat{\delta} (x_1^2 + \dots + x_{n-1}^2) - (n-1)\lambda^2) \right| \\ & \leq C r^{1-n-2\hat{\delta}}. \end{aligned}$$

Moreover, the metric g satisfies

$$|g - (1 + \alpha r^{2-n}) \bar{g}|_{\bar{g}} \leq C r^{2-n-2\hat{\delta}}$$

and

$$|\bar{D}(g - (1 + \alpha r^{2-n}) \bar{g})|_{\bar{g}} \leq C r^{1-n-2\hat{\delta}}.$$

Consequently, the mean curvature of N_λ^+ with respect to the metric g satisfies

$$\begin{aligned} & \left| H_{N_\lambda^+} + \frac{1}{2} (n-2)(n-1) \alpha \lambda r^{-n} \right. \\ & \quad \left. - (x_1^2 + \dots + x_{n-1}^2 + \lambda^2)^{-\frac{n+1+\hat{\delta}}{2}} (\hat{\delta} (x_1^2 + \dots + x_{n-1}^2) - (n-1)\lambda^2) \right| \\ & \leq C r^{1-n-2\hat{\delta}}. \end{aligned}$$

The function $\hat{\rho}$ satisfies

$$|\hat{\rho} - (1 + \hat{\beta} r^{2-n})| \leq C r^{2-n-2\hat{\delta}}$$

and

$$|\bar{D}(\hat{\rho} - (1 + \hat{\beta} r^{2-n}))|_{\bar{g}} \leq C r^{1-n-2\hat{\delta}}.$$

Consequently, the normal derivative of the function $\log \hat{\rho}$ along N_λ^+ satisfies

$$|\langle \nabla \log \hat{\rho}, \nu_{N_\lambda^+} \rangle + (n-2) \hat{\beta} \lambda r^{-n}| \leq C r^{1-n-2\hat{\delta}}.$$

Putting these facts together, we obtain

$$\begin{aligned} & \left| H_{N_\lambda^+} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^+} \rangle + \frac{1}{2} (n-2) ((n-1)\alpha + 2\hat{\beta}) \lambda r^{-n} \right. \\ & \quad \left. - (x_1^2 + \dots + x_{n-1}^2 + \lambda^2)^{-\frac{n+1+\hat{\delta}}{2}} (\hat{\delta} (x_1^2 + \dots + x_{n-1}^2) - (n-1)\lambda^2) \right| \\ & \leq C r^{1-n-2\hat{\delta}}. \end{aligned}$$

Recall that $(n-1)\alpha + 2\hat{\beta} < 0$ by (11).

It is convenient to divide N_λ^+ into two regions. In the region

$$N_\lambda^+ \cap \{\hat{\delta}(x_1^2 + \dots + x_{n-1}^2) \geq 4(n-1)\lambda^2\},$$

we have

$$\begin{aligned} & H_{N_\lambda^+} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^+} \rangle \\ & \geq (x_1^2 + \dots + x_{n-1}^2 + \lambda^2)^{-\frac{n+1+\hat{\delta}}{2}} (\hat{\delta}(x_1^2 + \dots + x_{n-1}^2) - (n-1)\lambda^2) \\ & \quad - C r^{1-n-2\hat{\delta}} \\ & \geq (x_1^2 + \dots + x_{n-1}^2 + \lambda^2)^{-\frac{n+1+\hat{\delta}}{2}} \left(\frac{1}{2} \hat{\delta}(x_1^2 + \dots + x_{n-1}^2) + (n-1)\lambda^2 \right) \\ & \quad - C r^{1-n-2\hat{\delta}}, \end{aligned}$$

and the expression on the right hand side is positive if λ is sufficiently large. Finally, in the region

$$N_\lambda^+ \cap \{\hat{\delta}(x_1^2 + \dots + x_{n-1}^2) \leq 4(n-1)\lambda^2\},$$

we have

$$H_{N_\lambda^+} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^+} \rangle \geq -\frac{1}{2}(n-2)((n-1)\alpha + 2\hat{\beta})\lambda r^{-n} - C r^{1-n-\hat{\delta}},$$

and the expression on the right hand side is positive if λ is sufficiently large. This completes the proof of Proposition 3.6.

Let us fix a large constant λ_0 with the following properties:

- For each $\lambda \geq \lambda_0$, we have

$$H_{N_\lambda^+} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^+} \rangle > 0$$

at each point on N_λ^+ .

- For each $\lambda \geq \lambda_0$, we have

$$H_{N_\lambda^-} + \langle \nabla \log \hat{\rho}, \nu_{N_\lambda^-} \rangle > 0$$

at each point on N_λ^- .

- The hypersurfaces $\{N_\lambda^+ : \lambda \in [\lambda_0, 4\lambda_0]\}$ form a foliation.
- The hypersurfaces $\{N_\lambda^- : \lambda \in [\lambda_0, 4\lambda_0]\}$ form a foliation.

Proposition 3.7. *There exists a large constant s_0 (depending on λ_0) with the following significance. If $s \geq s_0$ and $\lambda \in [\lambda_0, 4\lambda_0]$, then the hypersurface*

$$W_s = \{x_1^2 + \dots + x_{n-1}^2 = s^2\}$$

intersects N_λ^+ and N_λ^- transversally. Moreover, $\langle \nu_{N_\lambda^+}, \nu_{W_s} \rangle < 0$ at each point on $N_\lambda^+ \cap W_s$ and $\langle \nu_{N_\lambda^-}, \nu_{W_s} \rangle < 0$ at each point on $N_\lambda^- \cap W_s$. Here, ν_{W_s} denotes the outward-pointing unit normal vector to W_s .

Proof. We only prove the assertion for N_λ^+ . The proof for N_λ^- is analogous. Let $\bar{\nu}_{N_\lambda^+}$ denote the unit normal to N_λ^+ with respect to the Euclidean metric \bar{g} , and let $\bar{\nu}_{W_s}$ denote the outward-pointing unit normal vector to W_s with respect to the Euclidean metric. We compute

$$\langle \bar{\nu}_{N_\lambda^+}, \bar{\nu}_{W_s} \rangle_{\bar{g}} = -s (s^2 + \lambda^2)^{-\frac{n-1+\hat{\delta}}{2}} (1 + s^2 (s^2 + \lambda^2)^{1-n-\hat{\delta}})^{-\frac{1}{2}}$$

at each point on $N_\lambda^+ \cap W_s$. Hence, if s is sufficiently large (depending on λ_0), then

$$|\langle \bar{\nu}_{N_\lambda^+}, \bar{\nu}_{W_s} \rangle_{\bar{g}} + s^{2-n-\hat{\delta}}| \leq C s^{-n-\hat{\delta}}$$

at each point on $N_\lambda^+ \cap W_s$. Since the metric g satisfies

$$|g - (1 + \alpha r^{2-n}) \bar{g}|_{\bar{g}} \leq C r^{2-n-2\hat{\delta}}$$

and the Riemannian metric $(1 + \alpha r^{2-n}) \bar{g}$ is conformal to \bar{g} , we conclude that

$$|\langle \nu_{N_\lambda^+}, \nu_{W_s} \rangle_g - \langle \bar{\nu}_{N_\lambda^+}, \bar{\nu}_{W_s} \rangle_{\bar{g}}| \leq C s^{2-n-2\hat{\delta}}$$

at each point on $N_\lambda^+ \cap W_s$. Therefore, if s is sufficiently large (depending on λ_0), then we obtain

$$|\langle \nu_{N_\lambda^+}, \nu_{W_s} \rangle_g + s^{2-n-\hat{\delta}}| \leq C s^{2-n-2\hat{\delta}}$$

at each point on $N_\lambda^+ \cap W_s$. This completes the proof of Proposition 3.7.

Definition 3.8. For each $\lambda \in [\lambda_0, 4\lambda_0]$, we define an open domain $E_{\text{slab}, \lambda}$ by

$$E_{\text{slab}, \lambda} = E \setminus \left\{ |x_n| \geq \lambda - (n-3+\hat{\delta})^{-1} (x_1^2 + \dots + x_{n-1}^2 + \lambda^2)^{-\frac{n-3+\hat{\delta}}{2}} \right\}.$$

Note that $\partial E_{\text{slab}, \lambda} = N_\lambda^+ \cup N_\lambda^- \cup \partial E$.

We define a vector field X on $E_{\text{slab}, 4\lambda_0} \setminus E_{\text{slab}, \lambda_0}$ so that X is the unit normal vector field to the foliation N_λ^+ in the region

$$\left\{ \lambda_0 - (n-3+\hat{\delta})^{-1} (x_1^2 + \dots + x_{n-1}^2 + \lambda_0^2)^{-\frac{n-3+\hat{\delta}}{2}} \leq x_n < 4\lambda_0 - (n-3+\hat{\delta})^{-1} (x_1^2 + \dots + x_{n-1}^2 + 16\lambda_0^2)^{-\frac{n-3+\hat{\delta}}{2}} \right\}$$

and X is the unit normal vector field to the foliation N_λ^- in the region

$$\left\{ \lambda_0 - (n-3+\hat{\delta})^{-1} (x_1^2 + \dots + x_{n-1}^2 + \lambda_0^2)^{-\frac{n-3+\hat{\delta}}{2}} \leq -x_n < 4\lambda_0 - (n-3+\hat{\delta})^{-1} (x_1^2 + \dots + x_{n-1}^2 + 16\lambda_0^2)^{-\frac{n-3+\hat{\delta}}{2}} \right\}.$$

Note that $|X| = 1$ at each point in $E_{\text{slab}, 4\lambda_0} \setminus E_{\text{slab}, \lambda_0}$.

Proposition 3.9. *Let X denote the vector field constructed above. Then $\operatorname{div}_g(\hat{\rho}X) > 0$ at each point in $E_{\text{slab},4\lambda_0} \setminus E_{\text{slab},\lambda_0}$. Moreover, if $s \geq s_0$, then $\langle X, \nu_{W_s} \rangle < 0$ at each point in $(E_{\text{slab},4\lambda_0} \setminus E_{\text{slab},\lambda_0}) \cap W_s$.*

Proof. The first statement follows from Proposition 3.6. The second statement follows from Proposition 3.7.

3.4. Construction of a μ -bubble (possibly with singularities). In this subsection, we construct a suitable μ -bubble. The use of μ -bubbles in the study of scalar curvature was pioneered by Gromov [13].

In the following, Φ will denote the function constructed in Lemma 3.1.

Lemma 3.10. *We can find a small positive constant η_0 so that the following statement holds. The distance function $d_{(M,g)}(\cdot, \partial E)$ is smooth on the tubular neighborhood $E \cap \mathcal{N}_{(M,g)}(\partial E, 4\eta_0)$. Moreover, we have $\operatorname{div}(\hat{\rho}Y) > \hat{\rho}\Phi$ at each point in $E \cap \mathcal{N}_{(M,g)}(\partial E, 4\eta_0)$, where Y denotes the gradient of the function $-d_{(M,g)}(\cdot, \partial E)$.*

Proof. By Lemma 3.1, we know that $\Phi \rightarrow -\infty$ on the boundary ∂E . From this, the assertion follows. This completes the proof of Lemma 3.10.

Let us consider a sequence $s_j \rightarrow \infty$. For each $\lambda \in [\lambda_0, 4\lambda_0]$ and each j , we define an open domain $E_{\text{slab},\lambda}^{(j)}$ by

$$E_{\text{slab},\lambda}^{(j)} = E_{\text{slab},\lambda} \setminus \{x_1^2 + \dots + x_{n-1}^2 \geq s_j^2\}.$$

For each j , we consider a variational problem on the domain $E_{\text{slab},4\lambda_0}^{(j)}$.

Definition 3.11. Let η_0 be chosen as in Lemma 3.10. We denote by $\mathcal{Z}^{(j)}$ the collection of Caccioppoli sets $\Omega \subset E_{\text{slab},4\lambda_0}^{(j)}$ with the property that

$$\Omega \cap \mathcal{N}_{(M,g)}(\partial E, \eta_0) = \emptyset,$$

$$\Omega \cap \bigcup_{\lambda \in [3\lambda_0, 4\lambda_0)} N_\lambda^+ = \emptyset,$$

and

$$\bigcup_{\lambda \in [3\lambda_0, 4\lambda_0)} N_\lambda^- \setminus \{x_1^2 + \dots + x_{n-1}^2 \geq s_j^2\} \subset \Omega.$$

Note that the domain $E_{\text{slab},4\lambda_0}^{(j)} \cap \{x_n < -10\}$ belongs to $\mathcal{Z}^{(j)}$. In particular, $\mathcal{Z}^{(j)} \neq \emptyset$.

Definition 3.12. For each $\Omega \in \mathcal{Z}^{(j)}$, we define

$$\mathcal{F}^{(j)}(\Omega) = \int_{\partial^* \Omega \cap E_{\text{slab},4\lambda_0}^{(j)}} \hat{\rho} d\mathcal{H}^{n-1} - \int_{\Omega} \hat{\rho} \Phi d\mathcal{H}^n.$$

Since $\Phi \leq 0$ at each point in E , we have

$$\mathcal{F}^{(j)}(\Omega) \geq \int_{\partial^* \Omega \cap E_{\text{slab}, 4\lambda_0}^{(j)}} \hat{\rho} d\mathcal{H}^{n-1}$$

for each $\Omega \in \mathcal{Z}^{(j)}$.

Lemma 3.13. *Let η_0 be chosen as in Lemma 3.10. Let $\Omega \in \mathcal{Z}^{(j)}$, and suppose that Ω has smooth boundary in $E_{\text{slab}, 4\lambda_0}^{(j)}$. Let us choose $\tilde{\eta} \in (3\eta_0, 4\eta_0)$ so that $\partial\Omega$ is transversal to $\partial\mathcal{N}_{(M,g)}(\partial E, \tilde{\eta})$. Then $\mathcal{F}^{(j)}(\tilde{\Omega}) \leq \mathcal{F}^{(j)}(\Omega)$, where $\tilde{\Omega} = \Omega \setminus \mathcal{N}_{(M,g)}(\partial E, \tilde{\eta})$.*

Proof. Let Y denote the vector field constructed in Lemma 3.10. Then $|Y| = 1$ and $\text{div}_g(\hat{\rho}Y) > \hat{\rho}\Phi$ at each point in $E \cap \mathcal{N}_{(M,g)}(\partial E, 4\eta_0)$. Integrating this inequality over $\Omega \cap \mathcal{N}_{(M,g)}(\partial E, \tilde{\eta}) \subset E \cap \mathcal{N}_{(M,g)}(\partial E, 4\eta_0)$ gives

$$\begin{aligned} & \int_{\partial\Omega \cap \mathcal{N}_{(M,g)}(\partial E, \tilde{\eta})} \hat{\rho} d\mathcal{H}^{n-1} - \int_{\partial\Omega \cap \partial\mathcal{N}_{(M,g)}(\partial E, \tilde{\eta})} \hat{\rho} d\mathcal{H}^{n-1} \\ & \geq \int_{\partial\Omega \cap \mathcal{N}_{(M,g)}(\partial E, \tilde{\eta})} \hat{\rho} \langle Y, \nu_{\partial\Omega} \rangle d\mathcal{H}^{n-1} - \int_{\partial\Omega \cap \partial\mathcal{N}_{(M,g)}(\partial E, \tilde{\eta})} \hat{\rho} d\mathcal{H}^{n-1} \\ & = \int_{\Omega \cap \mathcal{N}_{(M,g)}(\partial E, \tilde{\eta})} \text{div}(\hat{\rho}Y) d\mathcal{H}^n \\ & \geq \int_{\Omega \cap \mathcal{N}_{(M,g)}(\partial E, \tilde{\eta})} \hat{\rho} \Phi d\mathcal{H}^n. \end{aligned}$$

This implies $\mathcal{F}^{(j)}(\tilde{\Omega}) \leq \mathcal{F}^{(j)}(\Omega)$. This completes the proof of Lemma 3.13.

Lemma 3.14. *Let $\Omega \in \mathcal{Z}^{(j)}$, and suppose that Ω has smooth boundary in $E_{\text{slab}, 4\lambda_0}^{(j)}$. Let us choose $\tilde{\lambda} \in (\lambda_0, \frac{3\lambda_0}{2})$ so that $\partial\Omega$ is transversal to $N_{\tilde{\lambda}}^+$. Then $\mathcal{F}^{(j)}(\tilde{\Omega}) \leq \mathcal{F}^{(j)}(\Omega)$, where*

$$\tilde{\Omega} = \Omega \setminus \bigcup_{\lambda \in (\tilde{\lambda}, 4\lambda_0)} N_{\lambda}^+.$$

Proof. Let X denote the vector field constructed in the previous subsection. Then $|X| = 1$, $\langle X, \nu_{W_{s_j}} \rangle < 0$ on $(E_{\text{slab}, 4\lambda_0} \setminus E_{\text{slab}, \lambda_0}) \cap W_{s_j}$, and $\text{div}_g(\hat{\rho}X) > 0$ in $E_{\text{slab}, 4\lambda_0} \setminus E_{\text{slab}, \lambda_0}$. We now integrate the inequality $\text{div}_g(\hat{\rho}X) > 0$ over

$$\Omega \cap \bigcup_{\lambda \in (\tilde{\lambda}, 4\lambda_0)} N_{\lambda}^+.$$

Consequently, the $\hat{\rho}$ -weighted area of

$$\partial\Omega \cap \bigcup_{\lambda \in (\tilde{\lambda}, 4\lambda_0)} N_{\lambda}^+$$

is bounded from below by the $\hat{\rho}$ -weighted area of $\Omega \cap N_{\tilde{\lambda}}^+$. Since $\Phi = 0$ on E_0 , it follows that $\mathcal{F}^{(j)}(\tilde{\Omega}) \leq \mathcal{F}^{(j)}(\Omega)$. This completes the proof of Lemma 3.14.

Lemma 3.15. *Let $\Omega \in \mathcal{Z}^{(j)}$, and suppose that Ω has smooth boundary in $E_{\text{slab},4\lambda_0}^{(j)}$. Let us choose $\tilde{\lambda} \in (\lambda_0, \frac{3\lambda_0}{2})$ so that $\partial\Omega$ is transversal to $N_{\tilde{\lambda}}^-$. Then $\mathcal{F}^{(j)}(\tilde{\Omega}) \leq \mathcal{F}^{(j)}(\Omega)$, where*

$$\tilde{\Omega} = \Omega \cup \left(\bigcup_{\lambda \in (\tilde{\lambda}, 4\lambda_0)} N_{\lambda}^- \setminus \{x_1^2 + \dots + x_{n-1}^2 \geq s_j^2\} \right).$$

Proof. Let X denote the vector field constructed in the previous subsection. Then $|X| = 1$, $\langle X, \nu_{W_{s_j}} \rangle < 0$ on $(E_{\text{slab},4\lambda_0} \setminus E_{\text{slab},\lambda_0}) \cap W_{s_j}$, and $\text{div}_g(\hat{\rho}X) > 0$ in $E_{\text{slab},4\lambda_0} \setminus E_{\text{slab},\lambda_0}$. We now integrate the inequality $\text{div}_g(\hat{\rho}X) > 0$ over

$$\bigcup_{\lambda \in (\tilde{\lambda}, 4\lambda_0)} N_{\lambda}^- \setminus (\Omega \cup \{x_1^2 + \dots + x_{n-1}^2 \geq s_j^2\}).$$

Consequently, the $\hat{\rho}$ -weighted area of

$$\partial\Omega \cap \bigcup_{\lambda \in (\tilde{\lambda}, 4\lambda_0)} N_{\lambda}^-$$

is bounded from below by the $\hat{\rho}$ -weighted area of $N_{\tilde{\lambda}}^- \setminus \Omega$. Since $\Phi = 0$ on E_0 , it follows that $\mathcal{F}^{(j)}(\tilde{\Omega}) \leq \mathcal{F}^{(j)}(\Omega)$. This completes the proof of Lemma 3.15.

Proposition 3.16 (Existence of a minimizer). *For each j , there exists a Caccioppoli set $\hat{\Omega}^{(j)} \in \mathcal{Z}^{(j)}$ which minimizes the functional $\mathcal{F}^{(j)}$. Moreover,*

$$(14) \quad \hat{\Omega}^{(j)} \cap \mathcal{N}_{(M,g)}(\partial E, 2\eta_0) = \emptyset,$$

$$(15) \quad \hat{\Omega}^{(j)} \cap \bigcup_{\lambda \in [2\lambda_0, 4\lambda_0)} N_{\lambda}^+ = \emptyset,$$

and

$$(16) \quad \bigcup_{\lambda \in [2\lambda_0, 4\lambda_0)} N_{\lambda}^- \setminus \{x_1^2 + \dots + x_{n-1}^2 \geq s_j^2\} \subset \hat{\Omega}^{(j)}.$$

In particular, the reduced boundary of $\hat{\Omega}^{(j)}$ is contained in the closure of $E_{\text{slab},2\lambda_0}^{(j)} \setminus \mathcal{N}_{(M,g)}(\partial E, 2\eta_0)$.

Proof. We fix an integer j . Let $\{\Omega^{(j,l)} : l = 1, 2, \dots\} \subset \mathcal{Z}^{(j)}$ be a minimizing sequence for the function $\mathcal{F}^{(j)}$. By Theorem 13.8 in [16], we

may assume that $\Omega^{(j,l)}$ has smooth boundary in $E_{\text{slab},4\lambda_0}^{(j)}$. In view of Lemma 3.13, Lemma 3.14, and Lemma 3.15, we may further assume that

$$(17) \quad \Omega^{(j,l)} \cap \mathcal{N}_{(M,g)}(\partial E, 3\eta_0) = \emptyset,$$

$$(18) \quad \Omega^{(j,l)} \cap \bigcup_{\lambda \in [\frac{3\lambda_0}{2}, 4\lambda_0)} N_\lambda^+ = \emptyset,$$

and

$$(19) \quad \bigcup_{\lambda \in [\frac{3\lambda_0}{2}, 4\lambda_0)} N_\lambda^- \setminus \{x_1^2 + \dots + x_{n-1}^2 \geq s_j^2\} \subset \Omega^{(j,l)}.$$

Clearly,

$$\sup_l \mathcal{H}^{n-1}(\partial^* \Omega^{(j,l)} \cap E_{\text{slab},4\lambda_0}^{(j)}) < \infty.$$

Thus, for each j , the sets $\{\Omega^{(j,l)} : l = 1, 2, \dots\}$ have bounded perimeter. We now invoke the compactness theorem for BV functions. Hence, we can find a Caccioppoli set $\hat{\Omega}^{(j)}$ such that, after passing to a subsequence if necessary, $1_{\Omega^{(j,l)}}$ converges to $1_{\hat{\Omega}^{(j)}}$ strongly in L^1 . The statements (14), (15), (16) follow from (17), (18), (19). Therefore, $\hat{\Omega}^{(j)} \in \mathcal{Z}^{(j)}$. Since the BV norm is lower semicontinuous with respect to L^1 convergence, it follows that

$$\mathcal{F}^{(j)}(\hat{\Omega}^{(j)}) \leq \limsup_{l \rightarrow \infty} \mathcal{F}^{(j)}(\Omega^{(j,l)}).$$

Thus, $\hat{\Omega}^{(j)}$ is the desired minimizer. This completes the proof of Proposition 3.16.

Lemma 3.17. *Let U be an open domain with smooth boundary which is contained in a compact subset of $E_{\text{slab},3\lambda_0} \setminus \mathcal{N}_{(M,g)}(\partial E, \frac{3\eta_0}{2})$. Then*

$$\int_{\partial^* \hat{\Omega}^{(j)} \cap U} \hat{\rho} d\mathcal{H}^{n-1} \leq \int_{\partial U} \hat{\rho} d\mathcal{H}^{n-1}.$$

Proof. Since U is contained in a compact subset of $E_{\text{slab},3\lambda_0} \setminus \mathcal{N}_{(M,g)}(\partial E, \frac{3\eta_0}{2})$, it follows that $\hat{\Omega}^{(j)} \setminus U \in \mathcal{Z}^{(j)}$. Using the minimization property of $\mathcal{F}^{(j)}$, we obtain

$$\mathcal{F}^{(j)}(\hat{\Omega}^{(j)}) \leq \mathcal{F}^{(j)}(\hat{\Omega}^{(j)} \setminus U).$$

This implies

$$\int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab},4\lambda_0}^{(j)} \cap U} \hat{\rho} d\mathcal{H}^{n-1} - \int_{\hat{\Omega}^{(j)} \cap U} \hat{\rho} \Phi d\mathcal{H}^n \leq \int_{\partial U} \hat{\rho} d\mathcal{H}^{n-1}.$$

Since $\Phi \leq 0$, the assertion follows. This completes the proof of Lemma 3.17.

Lemma 3.18. *We can find a large number s_* with the following significance. Suppose that j is sufficiently large so that $s_j \geq s_*$. Then*

$$\int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)} \cap \{\frac{1}{4} \bar{s}^2 < x_1^2 + \dots + x_{n-1}^2 < \bar{s}^2\}} \hat{\rho} d\mathcal{H}^{n-1} \leq C \bar{s}^{n-1}$$

for each $\bar{s} \in [s_*, s_j]$, where C is a uniform constant.

Proof. Let

$$U = E_{\text{slab}, \frac{5\lambda_0}{2}} \cap \left\{ \frac{1}{4} \bar{s}^2 < x_1^2 + \dots + x_{n-1}^2 < \bar{s}^2 \right\}.$$

Clearly, U is an open set which is contained in a compact subset of $E_{\text{slab}, 3\lambda_0} \setminus \mathcal{N}_{(M,g)}(\partial E, \frac{3\eta_0}{2})$. Arguing as in the proof of Lemma 3.17, we obtain

$$\int_{\partial^* \hat{\Omega}^{(j)} \cap U} \hat{\rho} d\mathcal{H}^{n-1} \leq \int_{\partial U} \hat{\rho} d\mathcal{H}^{n-1}.$$

From this, the assertion follows. This completes the proof of Lemma 3.18.

In the next step, we take a limit of the minimizers $\hat{\Omega}^{(j)}$ as $j \rightarrow \infty$. To do that, we use ideas from the work of Eichmair and Körber [10]. Using Lemma 3.17, we obtain local area bounds for $\hat{\Omega}^{(j)}$. We again invoke the compactness theorem for BV functions. After passing to a subsequence, we can find a Caccioppoli set $\hat{\Omega}$ such that $1_{\hat{\Omega}^{(j)}} \rightarrow 1_{\hat{\Omega}}$ in $L^1_{\text{loc}}(E_{\text{slab}, 4\lambda_0})$.

Definition 3.19. Let \hat{U} be an open set which is contained in a compact subset of $E_{\text{slab}, 3\lambda_0} \setminus \mathcal{N}_{(M,g)}(\partial E, \frac{3\eta_0}{2})$. Given a Caccioppoli set $\Omega \subset E_{\text{slab}, 4\lambda_0}$, we define

$$\mathcal{F}(\Omega; \hat{U}) = \int_{\partial^* \Omega \cap \hat{U}} \hat{\rho} d\mathcal{H}^{n-1} - \int_{\Omega \cap \hat{U}} \hat{\rho} \Phi d\mathcal{H}^n.$$

Since $\Phi \leq 0$ at each point in E , we obtain

$$\mathcal{F}(\Omega; \hat{U}) \geq \int_{\partial^* \Omega \cap \hat{U}} \hat{\rho} d\mathcal{H}^{n-1}.$$

Lemma 3.20. *Let \hat{U} be an open domain with smooth boundary which is contained in a compact subset of $E_{\text{slab}, 3\lambda_0} \setminus \mathcal{N}_{(M,g)}(\partial E, \frac{3\eta_0}{2})$. If $\Omega \subset E_{\text{slab}, 4\lambda_0}$ is a Caccioppoli set with the property that the symmetric difference $\Omega \Delta \hat{\Omega}$ is contained in a compact subset of \hat{U} , then $\mathcal{F}(\hat{\Omega}; \hat{U}) \leq \mathcal{F}(\Omega; \hat{U})$.*

Proof. For $s > 0$ sufficiently small, we define $U_s = \{x \in \hat{U} : d_{(M,g)}(x, \partial \hat{U}) > s\}$. Recall that

$$\int_{\hat{U}} |1_{\hat{\Omega}^{(j)}} - 1_{\hat{\Omega}}| \rightarrow 0$$

as $j \rightarrow \infty$. By the co-area formula, we can find a sequence of positive real numbers $\varepsilon_j \rightarrow 0$ and a sequence of positive real numbers $\hat{s}_j \rightarrow 0$ with the property that the boundary trace of $1_{\hat{\Omega}^{(j)}} - 1_{\hat{\Omega}}$ along $\partial U_{\hat{s}_j}$ has $L^1(\partial U_{\hat{s}_j})$ -norm less than ε_j .

By assumption, the symmetric difference $\Omega \Delta \hat{\Omega}$ is contained in a compact subset of \hat{U} . Hence, if j is sufficiently large, then the boundary trace of $1_{\hat{\Omega}^{(j)}} - 1_{\Omega}$ along $\partial U_{\hat{s}_j}$ has $L^1(\partial U_{\hat{s}_j})$ -norm less than ε_j .

If j is sufficiently large, then the set $(\hat{\Omega}^{(j)} \setminus U_{\hat{s}_j}) \cup (\Omega \cap U_{\hat{s}_j})$ belongs to $\mathcal{Z}^{(j)}$. Using the minimization property of $\mathcal{F}^{(j)}$, we obtain

$$\mathcal{F}^{(j)}(\hat{\Omega}^{(j)}) \leq \mathcal{F}^{(j)}((\hat{\Omega}^{(j)} \setminus U_{\hat{s}_j}) \cup (\Omega \cap U_{\hat{s}_j}))$$

if j is sufficiently large. This implies

$$\mathcal{F}(\hat{\Omega}^{(j)}; U_{\hat{s}_j}) \leq \mathcal{F}(\Omega; U_{\hat{s}_j}) + C\varepsilon_j$$

if j is sufficiently large. Finally, we send $j \rightarrow \infty$. Since the BV norm is lower semicontinuous with respect to L^1 convergence, we conclude that

$$\mathcal{F}(\hat{\Omega}; \tilde{U}) \leq \limsup_{j \rightarrow \infty} \mathcal{F}(\hat{\Omega}^{(j)}; U_{\hat{s}_j}) \leq \limsup_{j \rightarrow \infty} (\mathcal{F}(\Omega; U_{\hat{s}_j}) + C\varepsilon_j) \leq \mathcal{F}(\Omega; \hat{U})$$

for every open domain \tilde{U} with the property that the closure of \tilde{U} is contained in a compact subset of \hat{U} . This finally implies

$$\mathcal{F}(\hat{\Omega}; \hat{U}) \leq \mathcal{F}(\Omega; \hat{U}).$$

This completes the proof of Lemma 3.20.

Lemma 3.21. *If \bar{s} is sufficiently large, then the following holds. If $(\bar{x}_1, \dots, \bar{x}_{n-1})$ is a point in \mathbb{R}^{n-1} with $\sqrt{\bar{x}_1^2 + \dots + \bar{x}_{n-1}^2} = \bar{s}$, then*

$$\begin{aligned} & \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0} \cap \{(x_1 - \bar{x}_1)^2 + \dots + (x_{n-1} - \bar{x}_{n-1})^2 < \frac{1}{4} \bar{s}^2\}} \hat{\rho} d\mathcal{H}^{n-1} \\ & \leq (1 + o(1)) |B^{n-1}| \left(\frac{1}{2} \bar{s}\right)^{n-1}. \end{aligned}$$

Proof. Let

$$U = E_{\text{slab}, \frac{5\lambda_0}{2}} \cap \left\{ (x_1 - \bar{x}_1)^2 + \dots + (x_{n-1} - \bar{x}_{n-1})^2 < \frac{1}{4} \bar{s}^2 \right\}.$$

We can find an open domain \hat{U} with smooth boundary such that U is contained in a compact subset of \hat{U} and \hat{U} is contained in a compact subset of $E_{\text{slab}, 3\lambda_0} \setminus \mathcal{N}_{(M,g)}(\partial E, \frac{3\eta_0}{2})$. Using Lemma 3.20, we obtain

$$\mathcal{F}(\hat{\Omega}; \hat{U}) \leq \mathcal{F}(\hat{\Omega} \setminus U; \hat{U}).$$

This implies

$$\int_{\partial^* \hat{\Omega} \cap U} \hat{\rho} d\mathcal{H}^{n-1} \leq \int_{\partial U \setminus N_{3\lambda_0}^+} \hat{\rho} d\mathcal{H}^{n-1} \leq (1 + o(1)) |B^{n-1}| \left(\frac{1}{2} \bar{s}\right)^{n-1}.$$

This completes the proof of Lemma 3.21.

Definition 3.22. We denote by \mathcal{S} the singular set of $\hat{\Omega}$, and by $\Sigma = (\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}) \setminus \mathcal{S}$ the regular part of $\hat{\Omega}$. Note that Σ is a smooth (possibly disconnected) hypersurface in $E \setminus \mathcal{S}$, and $H_\Sigma + \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle = \Phi$ at each point on Σ .

In the remainder of this subsection, we show that the singular set \mathcal{S} is compact and Σ is asymptotically planar near infinity.

Proposition 3.23. *The singular set \mathcal{S} is compact. The second fundamental form of Σ satisfies $|h_\Sigma| \leq C r^{-1}$ near infinity. For every nonnegative integer m , the m -th order covariant derivative of the second fundamental form of Σ is bounded by $C(m) r^{-m-1}$ near infinity.*

Proof. This follows from Lemma 3.21 together with Allard's regularity theorem (see [2] or [22]) and standard interior estimates.

Corollary 3.24. *Near infinity, the hypersurface Σ can be written as a graph $x_n = u(x_1, \dots, x_{n-1})$. The function u is bounded. For every nonnegative integer m , the m -th order derivatives of u are bounded by $C(m) (x_1^2 + \dots + x_{n-1}^2)^{-\frac{m}{2}}$ near infinity.*

Proof. This follows by combining Proposition 3.23, the density bound in Lemma 3.21, and the fact that Σ is contained in a slab.

Proposition 3.25. *There exist real numbers c_0 and μ_0 such that*

$$\begin{aligned} & \left| u(x_1, \dots, x_{n-1}) - (c_0 + \mu_0 (x_1^2 + \dots + x_{n-1}^2)^{\frac{3-n}{2}}) \right| \\ & \leq O((x_1^2 + \dots + x_{n-1}^2)^{\frac{3-n-\hat{\delta}}{2}}), \end{aligned}$$

and we have analogous estimates for the higher derivatives of u .

Proof. The hypersurface Σ satisfies $H_\Sigma + \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle = 0$ near infinity. Hence, the restriction of the coordinate function x_n to Σ satisfies

$$(20) \quad \Delta_\Sigma x_n + \langle \nabla^\Sigma \log \hat{\rho}, \nabla^\Sigma x_n \rangle = \Delta x_n - (D^2 x_n)(\nu_\Sigma, \nu_\Sigma) + \langle \nabla \log \hat{\rho}, \nabla x_n \rangle$$

near infinity.

Since Σ is contained in a slab, we know that $|x_n| \leq C$ at each point in Σ . From this, we deduce that

$$(21) \quad |\Delta x_n| \leq O(r^{1-n-\hat{\delta}}),$$

$$(22) \quad |(D^2 x_n)(\nabla x_n, \nabla x_n)| \leq O(r^{1-n-\hat{\delta}}),$$

and

$$(23) \quad |\langle \nabla \log \hat{\rho}, \nabla x_n \rangle| \leq O(r^{1-n-\hat{\delta}})$$

along Σ . The inequality (22) implies

$$(24) \quad |(D^2 x_n)(\nu_\Sigma, \nu_\Sigma)| \leq O(r^{1-n-\hat{\delta}})$$

along Σ . Combining (20), (21), (23), and (24), we conclude that

$$|\Delta_{\Sigma}x_n + \langle \nabla^{\Sigma} \log \hat{\rho}, \nabla^{\Sigma}x_n \rangle| \leq O(r^{1-n-\hat{\delta}})$$

on Σ . Since $x_n = u(x_1, \dots, x_{n-1})$ at along Σ , it follows that

$$|\Delta_{\Sigma}u(x_1, \dots, x_{n-1}) + \langle \nabla^{\Sigma} \log \hat{\rho}, \nabla^{\Sigma}u(x_1, \dots, x_{n-1}) \rangle| \leq O(r^{1-n-\hat{\delta}})$$

on Σ . Therefore, the function u satisfies a linear PDE of the form

$$\begin{aligned} & \sum_{i,j=1}^{n-1} a_{ij}(x_1, \dots, x_{n-1}) \partial_i \partial_j u + \sum_{j=1}^n b_j(x_1, \dots, x_{n-1}) \partial_j u(x_1, \dots, x_{n-1}) \\ & = G(x_1, \dots, x_{n-1}) \end{aligned}$$

near infinity, where

$$\max_{1 \leq i,j \leq n-1} |a_{ij}(x_1, \dots, x_{n-1}) - \delta_{ij}| \leq O((x_1^2 + \dots + x_{n-1}^2)^{-\frac{1}{2}}),$$

$$\max_{1 \leq j \leq n-1} |b_j(x_1, \dots, x_{n-1})| \leq O((x_1^2 + \dots + x_{n-1}^2)^{-1}),$$

and

$$|G(x_1, \dots, x_{n-1})| \leq O((x_1^2 + \dots + x_{n-1}^2)^{\frac{1-n-\hat{\delta}}{2}}).$$

Moreover, we have analogous estimates for the higher derivatives of a_{ij} , b_j , and G . Since u is bounded, the assertion follows from standard results about linear PDE (see [17]). This completes the proof of Proposition 3.25.

3.5. The stability inequality for the μ -bubble. In this subsection, we show that $\hat{\Omega}$ satisfies a stability inequality. In the first step, we state the stability inequality for $\hat{\Omega}^{(j)}$. In the second step, we will pass to the limit as $j \rightarrow \infty$.

Proposition 3.26. *Let j be a large integer. Suppose that a is a real number and V is a smooth vector field on E such that $V = a \frac{\partial}{\partial x_n}$ in a neighborhood*

of $W_{s_j} \cap E_{\text{slab}, 3\lambda_0}$. Then

$$\begin{aligned}
 & \frac{1}{2} \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} \hat{\rho} \sum_{k=1}^{n-1} (\mathcal{L}_V \mathcal{L}_V g)(e_k, e_k) d\mathcal{H}^{n-1} \\
 & + \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} V(V(\hat{\rho})) d\mathcal{H}^{n-1} \\
 & - \frac{1}{2} \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_l) (\mathcal{L}_V g)(e_k, e_l) d\mathcal{H}^{n-1} \\
 & + \frac{1}{4} \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) (\mathcal{L}_V g)(e_l, e_l) d\mathcal{H}^{n-1} \\
 & + \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} V(\hat{\rho}) \sum_{k=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) d\mathcal{H}^{n-1} \\
 & \geq \int_{\hat{\Omega}^{(j)}} \operatorname{div}(\operatorname{div}(\hat{\rho} \Phi V) V) d\mathcal{H}^n.
 \end{aligned}$$

Proof. This follows from the fact that $\hat{\Omega}^{(j)}$ is a minimizer of the functional $\mathcal{F}^{(j)}$.

Proposition 3.27. *Suppose that a is a real number and V is a smooth vector field on E such that $V = a \frac{\partial}{\partial x_n}$ near infinity. Then*

$$\begin{aligned}
 & \frac{1}{2} \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}} \hat{\rho} \sum_{k=1}^{n-1} (\mathcal{L}_V \mathcal{L}_V g)(e_k, e_k) d\mathcal{H}^{n-1} \\
 & + \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}} V(V(\hat{\rho})) d\mathcal{H}^{n-1} \\
 & - \frac{1}{2} \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_l) (\mathcal{L}_V g)(e_k, e_l) d\mathcal{H}^{n-1} \\
 & + \frac{1}{4} \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) (\mathcal{L}_V g)(e_l, e_l) d\mathcal{H}^{n-1} \\
 & + \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}} V(\hat{\rho}) \sum_{k=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) d\mathcal{H}^{n-1} \\
 & \geq \int_{\hat{\Omega}} \operatorname{div}(\operatorname{div}(\hat{\rho} \Phi V) V) d\mathcal{H}^n.
 \end{aligned}$$

Proof. We first observe that

$$(25) \quad |\mathcal{L}_V g| \leq C |a| r^{1-n}, \quad |\mathcal{L}_V \mathcal{L}_V g| \leq C |a|^2 r^{-n}$$

and

$$(26) \quad |V(\hat{\rho})| \leq C |a| r^{1-n}, \quad |V(V(\hat{\rho}))| \leq C |a|^2 r^{-n}$$

near infinity.

Let us fix a smooth function $\chi : \mathbb{R} \rightarrow [0, 1]$ such that $\chi = 0$ on $(-\infty, \frac{1}{2}]$ and $\chi = 1$ on $[1, \infty)$. For $\bar{s} > 0$ sufficiently large, we define a smooth function $\Xi_{\bar{s}} : M \rightarrow \mathbb{R}$ by

$$\Xi_{\bar{s}}(x) = \begin{cases} \chi(\bar{s}^{-2}(x_1^2 + \dots + x_{n-1}^2)) & \text{for } x \in E_0 \\ 0 & \text{for } x \in M \setminus E_0. \end{cases}$$

In the following, we assume that j is chosen sufficiently large depending on \bar{s} . Using (25), (26), and Lemma 3.18, we can bound

$$\begin{aligned} & \frac{1}{2} \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} \Xi_{\bar{s}} \hat{\rho} \sum_{k=1}^{n-1} (\mathcal{L}_V \mathcal{L}_V g)(e_k, e_k) d\mathcal{H}^{n-1} \\ & + \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} \Xi_{\bar{s}} V(V(\hat{\rho})) d\mathcal{H}^{n-1} \\ & - \frac{1}{2} \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} \Xi_{\bar{s}} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_l) (\mathcal{L}_V g)(e_k, e_l) d\mathcal{H}^{n-1} \\ & + \frac{1}{4} \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} \Xi_{\bar{s}} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) (\mathcal{L}_V g)(e_l, e_l) d\mathcal{H}^{n-1} \\ & + \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} \Xi_{\bar{s}} V(\hat{\rho}) \sum_{k=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) d\mathcal{H}^{n-1} \\ & \leq C |a|^2 \bar{s}^{-1}, \end{aligned}$$

where C is independent of \bar{s} and j . Combining this inequality with Proposition 3.26, we obtain

$$\begin{aligned}
 & \frac{1}{2} \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} (1 - \Xi_{\bar{s}}) \hat{\rho} \sum_{k=1}^{n-1} (\mathcal{L}_V \mathcal{L}_V g)(e_k, e_k) d\mathcal{H}^{n-1} \\
 & + \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} (1 - \Xi_{\bar{s}}) V(V(\hat{\rho})) d\mathcal{H}^{n-1} \\
 & - \frac{1}{2} \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} (1 - \Xi_{\bar{s}}) \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_l) (\mathcal{L}_V g)(e_k, e_l) d\mathcal{H}^{n-1} \\
 & + \frac{1}{4} \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} (1 - \Xi_{\bar{s}}) \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) (\mathcal{L}_V g)(e_l, e_l) d\mathcal{H}^{n-1} \\
 & + \int_{\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}^{(j)}} (1 - \Xi_{\bar{s}}) V(\hat{\rho}) \sum_{k=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) d\mathcal{H}^{n-1} \\
 & \geq \int_{\hat{\Omega}^{(j)}} \operatorname{div}(\operatorname{div}(\hat{\rho} \Phi V) V) d\mathcal{H}^n - C |a|^2 \bar{s}^{-1},
 \end{aligned}$$

where C is independent of \bar{s} and j . In the next step, we send $j \rightarrow \infty$, keeping \bar{s} fixed. It follows from Theorem 21.14 in [16] that the $(n-1)$ -dimensional Hausdorff measure on $\partial^* \hat{\Omega}^{(j)} \cap E_{\text{slab}, 4\lambda_0}$ converges (in the sense of weak convergence of measures) to the $(n-1)$ -dimensional Hausdorff measure on $\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}$. In other words, there is no mass drop. Reshetnyak's continuity theorem (see e.g. [23]) now implies that the varifold associated with $\hat{\Omega}^{(j)}$ converges weakly to the varifold associated with $\hat{\Omega}$ as $j \rightarrow \infty$. This implies

$$\begin{aligned}
 & \frac{1}{2} \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}} (1 - \Xi_{\bar{s}}) \hat{\rho} \sum_{k=1}^{n-1} (\mathcal{L}_V \mathcal{L}_V g)(e_k, e_k) d\mathcal{H}^{n-1} \\
 & + \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}} (1 - \Xi_{\bar{s}}) V(V(\hat{\rho})) d\mathcal{H}^{n-1} \\
 & - \frac{1}{2} \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}} (1 - \Xi_{\bar{s}}) \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_l) (\mathcal{L}_V g)(e_k, e_l) d\mathcal{H}^{n-1} \\
 & + \frac{1}{4} \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}} (1 - \Xi_{\bar{s}}) \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) (\mathcal{L}_V g)(e_l, e_l) d\mathcal{H}^{n-1} \\
 & + \int_{\partial^* \hat{\Omega} \cap E_{\text{slab}, 4\lambda_0}} (1 - \Xi_{\bar{s}}) V(\hat{\rho}) \sum_{k=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) d\mathcal{H}^{n-1} \\
 & \geq \int_{\hat{\Omega}} \operatorname{div}(\operatorname{div}(\hat{\rho} \Phi V) V) d\mathcal{H}^n - C |a|^2 \bar{s}^{-1},
 \end{aligned}$$

where C is independent of \bar{s} . The assertion follows now by sending $\bar{s} \rightarrow \infty$. This completes the proof of Proposition 3.27.

Corollary 3.28. *Suppose that a is a real number and f is a smooth test function on Σ with the property that f vanishes near the singular set and $f = a \langle \frac{\partial}{\partial x_n}, \nu_\Sigma \rangle$ near infinity. Then*

$$\begin{aligned} & \int_{\Sigma} \hat{\rho} |\nabla^\Sigma f|^2 - \int_{\Sigma} \hat{\rho} \operatorname{Ric}(\nu_\Sigma, \nu_\Sigma) f^2 - \int_{\Sigma} \hat{\rho} |h_\Sigma|^2 f^2 \\ & + \int_{\Sigma} \hat{\rho} (D^2 \log \hat{\rho})(\nu_\Sigma, \nu_\Sigma) f^2 - \int_{\Sigma} \hat{\rho} \langle \nabla \Phi, \nu_\Sigma \rangle f^2 \geq 0. \end{aligned}$$

Proof. We can find a smooth vector field V on E such that $\langle V, \nu_\Sigma \rangle = f$ at each point on Σ , $V = 0$ in a neighborhood of the singular set, and $V = a \frac{\partial}{\partial x_n}$ near infinity. We define a vector field W on E by $W = D_V V$. Since $V = a \frac{\partial}{\partial x_n}$ near infinity, it follows that $W = a^2 D \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n}$ near infinity. This implies $|W| \leq C |a|^2 r^{1-n}$ near infinity.

In the next step, we define a tangential vector field Z along Σ by

$$Z = D_{V^{\tan}}^\Sigma(V^{\tan}) - \operatorname{div}_\Sigma(V^{\tan}) V^{\tan} + 2 \sum_{k=1}^{n-1} h_\Sigma(V^{\tan}, e_k) \langle V, \nu_\Sigma \rangle e_k.$$

Using Proposition 3.25, we obtain $|h_\Sigma| \leq C r^{1-n}$, $|V^{\tan}| \leq C |a| r^{2-n}$, and $|Z| \leq C |a|^2 r^{3-2n}$ near infinity. Proposition A.1 gives

$$\begin{aligned} & \hat{\rho} |\nabla^\Sigma f|^2 - \hat{\rho} (\operatorname{Ric}(\nu_\Sigma, \nu_\Sigma) + |h_\Sigma|^2) f^2 + (D^2 \hat{\rho})(\nu_\Sigma, \nu_\Sigma) f^2 - \hat{\rho}^{-1} \langle \nabla \hat{\rho}, \nu_\Sigma \rangle^2 f^2 \\ & + \operatorname{div}_\Sigma(\hat{\rho} W^{\tan}) - \operatorname{div}_\Sigma(\hat{\rho} Z) + \operatorname{div}_\Sigma(\langle V^{\tan}, \nabla^\Sigma \hat{\rho} \rangle V^{\tan}) \\ & - \hat{\rho} \langle \nabla \Phi, \nu_\Sigma \rangle f^2 + \operatorname{div}(\hat{\rho} \Phi V) \langle V, \nu_\Sigma \rangle + \operatorname{div}_\Sigma(\hat{\rho} \Phi \langle V, \nu_\Sigma \rangle V^{\tan}) \\ & = \frac{1}{2} \hat{\rho} \sum_{k=1}^{n-1} (\mathcal{L}_V \mathcal{L}_V g)(e_k, e_k) + V(V(\hat{\rho})) \\ & - \frac{1}{2} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_l) (\mathcal{L}_V g)(e_k, e_l) \\ & + \frac{1}{4} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) (\mathcal{L}_V g)(e_l, e_l) \\ & + V(\hat{\rho}) \sum_{k=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) \end{aligned}$$

at each point on Σ . Integrating this identity over Σ gives

$$\begin{aligned}
 & \int_{\Sigma} \hat{\rho} |\nabla^{\Sigma} f|^2 - \int_{\Sigma} \hat{\rho} (\text{Ric}(\nu_{\Sigma}, \nu_{\Sigma}) + |h_{\Sigma}|^2) f^2 \\
 & + \int_{\Sigma} (D^2 \hat{\rho})(\nu_{\Sigma}, \nu_{\Sigma}) f^2 - \int_{\Sigma} \hat{\rho}^{-1} \langle \nabla \hat{\rho}, \nu_{\Sigma} \rangle^2 f^2 \\
 & - \int_{\Sigma} \hat{\rho} \langle \nabla \Phi, \nu_{\Sigma} \rangle f^2 + \int_{\Sigma} \text{div}(\hat{\rho} \Phi V) \langle V, \nu_{\Sigma} \rangle \\
 & = \frac{1}{2} \int_{\Sigma} \hat{\rho} \sum_{k=1}^{n-1} (\mathcal{L}_V \mathcal{L}_V g)(e_k, e_k) + \int_{\Sigma} V(V(\hat{\rho})) \\
 & - \frac{1}{2} \int_{\Sigma} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_l) (\mathcal{L}_V g)(e_k, e_l) \\
 & + \frac{1}{4} \int_{\Sigma} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) (\mathcal{L}_V g)(e_l, e_l) \\
 & + \int_{\Sigma} V(\hat{\rho}) \sum_{k=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k).
 \end{aligned}$$

The assertion follows now from Proposition 3.27. This completes the proof of Corollary 3.28.

Remark 3.29. From now on, we will no longer work with the domain $\hat{\Omega}$, but we will work exclusively with the hypersurface Σ . The hypersurface Σ may be disconnected. If Σ is disconnected, then we focus on the connected component of Σ that contains the asymptotically planar end. We discard all the other connected components of Σ .

3.6. A generalized Schoen-Yau identity. In this subsection, we prove a generalized Schoen-Yau identity on Σ .

Proposition 3.30. *We have*

$$\begin{aligned}
 & \text{Ric}(\nu_{\Sigma}, \nu_{\Sigma}) + |h_{\Sigma}|^2 - (D^2 \log \hat{\rho})(\nu_{\Sigma}, \nu_{\Sigma}) + \langle \nabla \Phi, \nu_{\Sigma} \rangle \\
 & + \frac{1}{2} \left(R_{\Sigma} - 2 \Delta_{\Sigma} \log \hat{\rho} - \frac{n}{n+1} |\nabla^{\Sigma} \log \hat{\rho}|^2 \right) \geq \hat{Q}
 \end{aligned}$$

at each point on Σ . Here, \hat{Q} is defined as in Lemma 3.1.

Proof. The hypersurface Σ satisfies

$$(27) \quad H_{\Sigma} + \langle \nabla \log \hat{\rho}, \nu_{\Sigma} \rangle = \Phi.$$

The Gauss equations imply that

$$(28) \quad \frac{1}{2} R - \text{Ric}(\nu_{\Sigma}, \nu_{\Sigma}) - \frac{1}{2} R_{\Sigma} + \frac{1}{2} H_{\Sigma}^2 - \frac{1}{2} |h_{\Sigma}|^2 = 0$$

at each point on Σ . Moreover,

$$(29) \quad \Delta_\Sigma \log \hat{\rho} = \Delta \log \hat{\rho} - (D^2 \log \hat{\rho})(\nu_\Sigma, \nu_\Sigma) - H_\Sigma \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle$$

and

$$(30) \quad |\nabla^\Sigma \log \hat{\rho}|^2 = |\nabla \log \hat{\rho}|^2 - \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle^2.$$

This gives

$$\begin{aligned} & \text{Ric}(\nu_\Sigma, \nu_\Sigma) + |h_\Sigma|^2 - (D^2 \log \hat{\rho})(\nu_\Sigma, \nu_\Sigma) + \langle \nabla \Phi, \nu_\Sigma \rangle \\ & + \frac{1}{2} \left(R_\Sigma - 2 \Delta_\Sigma \log \hat{\rho} - \frac{n}{n+1} |\nabla^\Sigma \log \hat{\rho}|^2 \right) \\ & = \frac{1}{2} R + \frac{1}{2} H_\Sigma^2 + \frac{1}{2} |h_\Sigma|^2 + \langle \nabla \Phi, \nu_\Sigma \rangle \\ & - \Delta_\Sigma \log \hat{\rho} - (D^2 \log \hat{\rho})(\nu_\Sigma, \nu_\Sigma) - \frac{n}{2(n+1)} |\nabla^\Sigma \log \hat{\rho}|^2 \\ & = \frac{1}{2} R + \frac{1}{2} H_\Sigma^2 + \frac{1}{2} |h_\Sigma|^2 + \langle \nabla \Phi, \nu_\Sigma \rangle \\ & - \Delta \log \hat{\rho} + H_\Sigma \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle \\ & - \frac{n}{2(n+1)} |\nabla \log \hat{\rho}|^2 + \frac{n}{2(n+1)} \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle^2 \\ & = \frac{1}{2} R + \frac{1}{4} \Phi^2 + \frac{1}{2} \left(|h_\Sigma|^2 - \frac{1}{n-1} H_\Sigma^2 \right) + \langle \nabla \Phi, \nu_\Sigma \rangle \\ & - (\Delta \log \rho + \Delta \log \hat{v}) - \frac{n}{2(n+1)} |\nabla \log \rho + \nabla \log \hat{v}|^2 \\ & + \frac{n+1}{4(n-1)} \left(H_\Sigma + \frac{n-1}{n+1} \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle \right)^2 \\ & = \frac{1}{2} Q + \frac{1}{4} \Phi^2 + \frac{1}{2} \left(|h_\Sigma|^2 - \frac{1}{n-1} H_\Sigma^2 \right) + \langle \nabla \Phi, \nu_\Sigma \rangle \\ & + \frac{n+2}{2(n+1)} \left| \nabla \log \hat{v} + \frac{1}{n+2} \nabla \log \rho \right|^2 \\ & + \frac{n+1}{4(n-1)} \left(H_\Sigma + \frac{n-1}{n+1} \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle \right)^2 \\ & \geq \frac{1}{2} Q + \frac{1}{4} \Phi^2 - |\nabla \Phi| \end{aligned}$$

at each point on Σ . The first equality follows from (28). The second equality follows from (29) and (30). The third equality follows from (27) and the identity $\hat{\rho} = \rho \hat{v}$. The fourth equality follows from (13).

Finally, $Q + \frac{1}{2} \Phi^2 - 2 |\nabla \Phi| \geq 2 \hat{Q}$ by Lemma 3.1. This completes the proof of Proposition 3.30.

Corollary 3.31. *Let f be a smooth test function on Σ with the property that f vanishes near the singular set and f is constant near infinity. Then*

$$\begin{aligned} & \int_{\Sigma} \hat{\rho} |\nabla^{\Sigma} f|^2 + \frac{1}{2} \int_{\Sigma} \hat{\rho} \left(R_{\Sigma} - 2 \Delta_{\Sigma} \log \hat{\rho} - \frac{n}{n+1} |\nabla^{\Sigma} \log \hat{\rho}|^2 \right) f^2 \\ & \geq \int_{\Sigma} \hat{\rho} \hat{Q} f^2. \end{aligned}$$

Proof. Using Proposition 3.28 and a standard approximation argument, we obtain

$$\begin{aligned} & \int_{\Sigma} \hat{\rho} |\nabla^{\Sigma} f|^2 - \int_{\Sigma} \hat{\rho} \text{Ric}(\nu_{\Sigma}, \nu_{\Sigma}) f^2 - \int_{\Sigma} \hat{\rho} |h_{\Sigma}|^2 f^2 \\ & + \int_{\Sigma} \hat{\rho} (D^2 \log \hat{\rho})(\nu_{\Sigma}, \nu_{\Sigma}) f^2 - \int_{\Sigma} \hat{\rho} \langle \nabla \Phi, \nu_{\Sigma} \rangle f^2 \geq 0. \end{aligned}$$

The assertion follows now from Proposition 3.30. This completes the proof of Corollary 3.31.

3.7. A function on ambient space that blows up at the singular set at a controlled rate. In this subsection, we construct a function that will allow us to apply a conformal blow-up technique in the spirit of Bi-Hao-He-Shi-Zhu [4].

Throughout this subsection, we assume that the singular set \mathcal{S} is non-empty. In this case, $n \geq 8$. Note that \mathcal{S} is a compact subset of E . In particular, \mathcal{S} has positive distance from the boundary ∂E . Let us fix a positive real number t_* so that $\sqrt{t_*} < \frac{1}{2} d_{(M,g)}(\mathcal{S}, \partial E)$ and $\sqrt{t_*} < \frac{1}{2} \text{inj}_{(M,g)}(p)$ for each point $p \in \mathcal{S}$.

Let us fix a nonnegative smooth function $\zeta : (0, \infty) \rightarrow \mathbb{R}$ such that

$$\zeta(t) = \frac{2}{n-3} t^{\frac{3-n}{2}} + \frac{4}{2n-7} t^{\frac{7-2n}{4}}$$

for $t \in (0, \frac{t_*}{4}]$ and

$$\zeta(t) = 0$$

for $t \in [t_*, \infty)$. In the following, Φ will denote the function constructed in Lemma 3.1.

Lemma 3.32. *We can find a small constant $t_0 \in (0, \frac{t_*}{4})$ with the following significance. Let p be a point in \mathcal{S} . Let us define a smooth function $\psi : E \setminus \{p\} \rightarrow \mathbb{R}$ by*

$$\psi(x) = \zeta(d_{(M,g)}(p, x)^2)$$

for all $x \in E \setminus \{p\}$. Then

$$\begin{aligned} & \Delta \psi - (D^2 \psi)(\xi, \xi) + \frac{n-3}{n+1} \langle \nabla \log \hat{\rho}, \nabla \psi \rangle \\ & - \left(\Phi - \frac{4}{n+1} \langle \nabla \log \hat{\rho}, \xi \rangle \right) \langle \nabla \psi, \xi \rangle \leq -\frac{1}{2} d_{(M,g)}(p, x)^{\frac{3-2n}{2}} \end{aligned}$$

for each point $x \in \mathcal{B}_{(M,g)}(p, \sqrt{t_0})$ and every unit vector $\xi \in T_x M$. Moreover,

$$\begin{aligned} & \Delta\psi - (D^2\psi)(\xi, \xi) + \frac{n-3}{n+1} \langle \nabla \log \hat{\rho}, \nabla\psi \rangle \\ & - \left(\Phi - \frac{4}{n+1} \langle \nabla \log \hat{\rho}, \xi \rangle \right) \langle \nabla\psi, \xi \rangle \leq C \end{aligned}$$

for each point $x \in E \setminus \{p\}$ and every unit vector $\xi \in T_x M$.

Proof. We define a function $\tau : E \rightarrow \mathbb{R}$ by $\tau(x) = d_{(M,g)}(p, x)^2$ for all $x \in E$. Since $\sqrt{t_*} < \frac{1}{2} \text{inj}_{(M,g)}(p)$, the function τ is smooth in $\mathcal{B}_{(M,g)}(p, \sqrt{t_*})$. The gradient and Hessian of τ satisfy

$$|d\tau|^2 = 4\tau$$

and

$$D^2\tau \geq 2(1 - C_0\tau)g$$

at each point $x \in \mathcal{B}_{(M,g)}(p, \sqrt{t_*})$. We next observe that

$$d\psi = -\left(\tau^{\frac{1-n}{2}} + \tau^{\frac{3-2n}{4}}\right) d\tau$$

and

$$\begin{aligned} D^2\psi &= -\left(\tau^{\frac{1-n}{2}} + \tau^{\frac{3-2n}{4}}\right) D^2\tau \\ &+ \frac{1}{4} \left(2(n-1)\tau^{-\frac{n+1}{2}} + (2n-3)\tau^{-\frac{2n+1}{4}}\right) d\tau \otimes d\tau \end{aligned}$$

at each point $x \in \mathcal{B}_{(M,g)}(p, \frac{\sqrt{t_*}}{2})$. Consequently,

$$\begin{aligned} & \Delta\psi - (D^2\psi)(\xi, \xi) \\ &= -\left(\tau^{\frac{1-n}{2}} + \tau^{\frac{3-2n}{4}}\right) (\Delta\tau - (D^2\tau)(\xi, \xi)) \\ &+ \frac{1}{4} \left(2(n-1)\tau^{-\frac{n+1}{2}} + (2n-3)\tau^{-\frac{2n+1}{4}}\right) (|\nabla\tau|^2 - \langle \nabla\tau, \xi \rangle^2) \end{aligned}$$

at each point $x \in \mathcal{B}_{(M,g)}(p, \frac{\sqrt{t_*}}{2})$. Using the estimate $\Delta\tau - (D^2\tau)(\xi, \xi) \geq 2(n-1)(1 - C_0\tau)$ and the inequality $|\nabla\tau|^2 - \langle \nabla\tau, \xi \rangle^2 \leq |\nabla\tau|^2 = 4\tau$, we obtain

$$\begin{aligned} \Delta\psi - (D^2\psi)(\xi, \xi) &\leq -2(n-1)\left(\tau^{\frac{1-n}{2}} + \tau^{\frac{3-2n}{4}}\right)(1 - C_0\tau) \\ &+ \left(2(n-1)\tau^{-\frac{n+1}{2}} + (2n-3)\tau^{-\frac{2n+1}{4}}\right)\tau \\ &= -\tau^{\frac{3-2n}{4}} + 2(n-1)C_0\left(\tau^{\frac{3-n}{2}} + \tau^{\frac{7-2n}{4}}\right) \end{aligned}$$

at each point $x \in \mathcal{B}_{(M,g)}(p, \frac{\sqrt{t_*}}{2})$. Since $\sqrt{t_*} < \frac{1}{2} d_{(M,g)}(\mathcal{S}, \partial E)$, the function Φ is uniformly bounded on $\mathcal{B}_{(M,g)}(p, \sqrt{t_*})$. This implies

$$\begin{aligned} & \frac{n-3}{n+1} \langle \nabla \log \hat{\rho}, \nabla\psi \rangle - \left(\Phi - \frac{4}{n+1} \langle \nabla \log \hat{\rho}, \xi \rangle \right) \langle \nabla\psi, \xi \rangle \\ & \leq C_1 |d\psi| = 2C_1 \left(\tau^{\frac{2-n}{2}} + \tau^{\frac{5-2n}{4}}\right) \end{aligned}$$

at each point $x \in \mathcal{B}_{(M,g)}(p, \frac{\sqrt{t_*}}{2})$. Putting these facts together, we conclude that

$$\begin{aligned} & \Delta\psi - (D^2\psi)(\xi, \xi) + \frac{n-3}{n+1} \langle \nabla \log \hat{\rho}, \nabla\psi \rangle \\ & - \left(\Phi - \frac{4}{n+1} \langle \nabla \log \hat{\rho}, \xi \rangle \right) \langle \nabla\psi, \xi \rangle \\ & \leq -\tau^{\frac{3-2n}{4}} + 2(n-1)C_0 \left(\tau^{\frac{3-n}{2}} + \tau^{\frac{7-2n}{4}} \right) \\ & + 2C_1 \left(\tau^{\frac{2-n}{2}} + \tau^{\frac{5-2n}{4}} \right) \end{aligned}$$

at each point $x \in \mathcal{B}_{(M,g)}(p, \frac{\sqrt{t_*}}{2})$. Hence, if we choose t_0 sufficiently small, then the first statement holds. Moreover, the second statement holds in the ball $\mathcal{B}_{(M,g)}(p, \frac{\sqrt{t_*}}{2})$. Finally, it is easy to see that the second statement holds on the set $E \setminus \mathcal{B}_{(M,g)}(p, \frac{\sqrt{t_*}}{2})$. This completes the proof of Lemma 3.32.

In the next step, we need an estimate for the Minkowski dimension of the singular set \mathcal{S} .

Theorem 3.33 (cf. J. Cheeger, A. Naber [7], Theorem 5.8). *The singular set \mathcal{S} has Minkowski dimension at most $n - 8$.*

The bound for the Minkowski dimension of the singular set was originally proved by Cheeger and Naber [7] for area-minimizing currents in codimension 1. Their arguments rely on the monotonicity formula and can be generalized to the setting of μ -bubbles (see [1],[11]).

For each positive integer l , we define $t_l = 4^{-l} t_0$. For each positive integer l , we choose a finite subset $\mathcal{S}^{(l)} \subset \mathcal{S}$ so that the balls $\{\mathcal{B}_{(M,g)}(p, \sqrt{t_l}) : p \in \mathcal{S}^{(l)}\}$ are disjoint, and so that $\mathcal{S}^{(l)}$ is maximal with respect to this property. Clearly,

$$\sum_{p \in \mathcal{S}^{(l)}} \text{vol}_g(\mathcal{B}_{(M,g)}(p, \sqrt{t_l})) \leq \text{vol}_g(\mathcal{N}_{(M,g)}(\mathcal{S}, \sqrt{t_l})).$$

Consequently, the cardinality of $\mathcal{S}^{(l)}$ is bounded from above by

$$|\mathcal{S}^{(l)}| \leq C t_l^{-\frac{n}{2}} \text{vol}_g(\mathcal{N}_{(M,g)}(\mathcal{S}, \sqrt{t_l})).$$

On the other hand, since \mathcal{S} has Minkowski dimension at most $n - 8$, we know that

$$\text{vol}_g(\mathcal{N}_{(M,g)}(\mathcal{S}, \sqrt{t_l})) \leq C(\mu) t_l^{\frac{8-\mu}{2}}$$

for each $\mu > 0$. Putting these facts together, we obtain

$$|\mathcal{S}^{(l)}| \leq C(\mu) t_l^{\frac{8-n-\mu}{2}}$$

for each $\mu > 0$. In particular, $\sum_{l=1}^{\infty} t_l^{\frac{n-5}{2}} |\mathcal{S}^{(l)}| < \infty$.

We now define a function $\Psi : E \setminus \mathcal{S} \rightarrow \mathbb{R}$ by

$$(31) \quad \Psi(x) = \sum_{l=1}^{\infty} \sum_{p \in \mathcal{S}^{(l)}} t_l^{\frac{n-5}{2}} \zeta(d_{(M,g)}(p, x)^2)$$

for all points $x \in E \setminus \mathcal{S}$. Since $\sum_{l=1}^{\infty} t_l^{\frac{n-5}{2}} |\mathcal{S}^{(l)}| < \infty$, the series converges and the function Ψ is well-defined. Moreover, using the fact that $\sum_{l=1}^{\infty} t_l^{\frac{n-5}{2}} |\mathcal{S}^{(l)}| < \infty$, it is easy to see that Ψ is a smooth function on $E \setminus \mathcal{S}$. Indeed, for every nonnegative integer m , we can bound

$$\begin{aligned} |D^m \Psi| &\leq C(m) \sum_{l=1}^{\infty} \sum_{p \in \mathcal{S}^{(l)}} t_l^{\frac{n-5}{2}} d_{(M,g)}(p, x)^{3-n-m} \\ &\leq C(m) \sum_{l=1}^{\infty} t_l^{\frac{n-5}{2}} |\mathcal{S}^{(l)}| d_{(M,g)}(x, \mathcal{S})^{3-n-m}, \end{aligned}$$

where D^m denotes the covariant derivative of order m with respect to the metric g .

Proposition 3.34. *Suppose that x is a point in $E \setminus \mathcal{S}$ satisfying $d_{(M,g)}(x, \mathcal{S}) < \frac{1}{4} \sqrt{t_0}$. Then $\Psi(x) \geq \frac{1}{2(n-3)} 3^{3-n} d_{(M,g)}(x, \mathcal{S})^{-2}$.*

Proof. Since $d_{(M,g)}(x, \mathcal{S}) < \frac{1}{4} \sqrt{t_0}$, we can find an integer $l_0 \geq 2$ such that $\sqrt{t_{l_0+1}} \leq d_{(M,g)}(x, \mathcal{S}) < \sqrt{t_{l_0}}$. We can find a point $q \in \mathcal{S}$ such that $d_{(M,g)}(x, q) \leq \sqrt{t_{l_0}}$. In view of the maximality of $\mathcal{S}^{(l_0)}$, we can find a point $p_0 \in \mathcal{S}^{(l_0)}$ such that $\mathcal{B}_{(M,g)}(p_0, \sqrt{t_{l_0}}) \cap \mathcal{B}_{(M,g)}(q, \sqrt{t_{l_0}}) \neq \emptyset$. Using the triangle inequality, we obtain $d_{(M,g)}(p_0, q) \leq 2\sqrt{t_{l_0}}$. This implies

$$d_{(M,g)}(p_0, x) \leq d_{(M,g)}(p_0, q) + d_{(M,g)}(x, q) \leq 3\sqrt{t_{l_0}}.$$

Since $l_0 \geq 2$, it follows that $d_{(M,g)}(p_0, x) \leq \sqrt{t_0} \leq \frac{\sqrt{t_*}}{2}$. Thus, we conclude that

$$\zeta(d_{(M,g)}(p_0, x)^2) \geq \frac{2}{n-3} d_{(M,g)}(p_0, x)^{3-n} \geq \frac{2}{n-3} 3^{3-n} t_{l_0}^{\frac{3-n}{2}}.$$

This finally implies

$$\begin{aligned} \Psi(x) &\geq t_{l_0}^{\frac{n-5}{2}} \zeta(d_{(M,g)}(p_0, x)^2) \\ &\geq \frac{2}{n-3} 3^{3-n} t_{l_0}^{-1} \\ &\geq \frac{1}{2(n-3)} 3^{3-n} d_{(M,g)}(x, \mathcal{S})^{-2}. \end{aligned}$$

This completes the proof of Proposition 3.34.

Proposition 3.35. *Suppose that x is a point in $E \setminus \mathcal{S}$ satisfying $d_{(M,g)}(x, \mathcal{S}) < \frac{1}{4} \sqrt{t_0}$, and $\xi \in T_x M$ is a unit vector. If $d_{(M,g)}(x, \mathcal{S})$ is sufficiently small, then*

$$\begin{aligned} & \Delta \Psi - (D^2 \Psi)(\xi, \xi) + \frac{n-3}{n+1} \langle \nabla \log \hat{\rho}, \nabla \Psi \rangle \\ & - \left(\Phi - \frac{4}{n+1} \langle \nabla \log \hat{\rho}, \xi \rangle \right) \langle \nabla \Psi, \xi \rangle < 0 \end{aligned}$$

at the point x .

Proof. Since $d_{(M,g)}(x, \mathcal{S}) < \frac{1}{4} \sqrt{t_0}$, we can find an integer $l_0 \geq 2$ such that $\sqrt{t_{l_0+1}} \leq d_{(M,g)}(x, \mathcal{S}) < \sqrt{t_{l_0}}$. As above, we can find a point $p_0 \in \mathcal{S}^{(l_0)}$ such that $d_{(M,g)}(p_0, x) \leq 3\sqrt{t_{l_0}}$. Using Lemma 3.32, we obtain

$$\begin{aligned} & \Delta \Psi - (D^2 \Psi)(\xi, \xi) + \frac{n-3}{n+1} \langle \nabla \log \hat{\rho}, \nabla \Psi \rangle \\ & - \left(\Phi - \frac{4}{n+1} \langle \nabla \log \hat{\rho}, \xi \rangle \right) \langle \nabla \Psi, \xi \rangle \\ & \leq -\frac{1}{2} t_{l_0}^{\frac{n-5}{2}} d_{(M,g)}(p_0, x)^{\frac{3-2n}{2}} + C \sum_{l=1}^{\infty} t_l^{\frac{n-5}{2}} |\mathcal{S}^{(l)}| \\ & \leq -\frac{1}{2} 3^{\frac{3-2n}{2}} t_{l_0}^{-\frac{7}{4}} + C \sum_{l=1}^{\infty} t_l^{\frac{n-5}{2}} |\mathcal{S}^{(l)}|. \end{aligned}$$

Since $\sum_{l=1}^{\infty} t_l^{\frac{n-5}{2}} |\mathcal{S}^{(l)}| < \infty$, the expression on the right hand side is negative if l_0 is sufficiently large. This completes the proof of Proposition 3.35.

Corollary 3.36. *We can find a small positive real number ε_0 with the property that*

$$\begin{aligned} & \Delta \Psi - (D^2 \Psi)(\xi, \xi) + \frac{n-3}{n+1} \langle \nabla \log \hat{\rho}, \nabla \Psi \rangle \\ & - \left(\Phi - \frac{4}{n+1} \langle \nabla \log \hat{\rho}, \xi \rangle \right) \langle \nabla \Psi, \xi \rangle \leq \frac{n-3}{n+1} \varepsilon_0^{-1} \hat{Q} \end{aligned}$$

for each point $x \in E \setminus \mathcal{S}$ and every unit vector $\xi \in T_x M$.

Proof. The function Ψ vanishes near ∂E and near infinity. Moreover, Proposition 3.35 implies that

$$\begin{aligned} & \Delta \Psi - (D^2 \Psi)(\xi, \xi) + \frac{n-3}{n+1} \langle \nabla \log \hat{\rho}, \nabla \Psi \rangle \\ & - \left(\Phi - \frac{4}{n+1} \langle \nabla \log \hat{\rho}, \xi \rangle \right) \langle \nabla \Psi, \xi \rangle < 0 \end{aligned}$$

if $d_{(M,g)}(x, \mathcal{S})$ is sufficiently small and $\xi \in T_x M$ is a unit vector. Since the function \hat{Q} is strictly positive, the assertion follows. This completes the proof of Corollary 3.36.

Corollary 3.37. *Let ε_0 be chosen as in Corollary 3.36. Then*

$$\Delta_\Sigma \Psi + \frac{n-3}{n+1} \langle \nabla^\Sigma \log \hat{\rho}, \nabla^\Sigma \Psi \rangle \leq \frac{n-3}{n+1} \varepsilon_0^{-1} \hat{Q}$$

at each point on Σ .

Proof. The hypersurface Σ satisfies

$$H_\Sigma + \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle = \Phi.$$

This implies

$$\begin{aligned} & \Delta_\Sigma \Psi + \frac{n-3}{n+1} \langle \nabla^\Sigma \log \hat{\rho}, \nabla^\Sigma \Psi \rangle \\ &= \Delta \Psi - (D^2 \Psi)(\nu_\Sigma, \nu_\Sigma) + \frac{n-3}{n+1} \langle \nabla \log \hat{\rho}, \nabla \Psi \rangle \\ & \quad - \left(\Phi - \frac{4}{n+1} \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle \right) \langle \nabla \Psi, \nu_\Sigma \rangle. \end{aligned}$$

If we apply Proposition 3.35 with $\xi = \nu_\Sigma$, the assertion follows. This completes the proof of Corollary 3.37.

3.8. The conformal blow-up procedure and the conclusion of the inductive step. In this subsection, we complete the inductive step and conclude the proof of Theorem 1.5. Throughout this subsection, we denote by \hat{g} the induced metric on Σ . The metric \hat{g} satisfies

$$(32) \quad \hat{g} = dx_1 \otimes dx_1 + \dots + dx_{n-1} \otimes dx_{n-1} + O((x_1^2 + \dots + x_{n-1}^2)^{\frac{2-n}{2}})$$

near infinity. Moreover, the restriction of the function $\hat{\rho}$ to Σ satisfies

$$(33) \quad \hat{\rho}|_\Sigma = 1 + O((x_1^2 + \dots + x_{n-1}^2)^{\frac{2-n}{2}})$$

near infinity.

We first consider the case that the singular set \mathcal{S} is empty.

Proposition 3.38. *If $\mathcal{S} = \emptyset$, then $(\Sigma, \hat{g}, \hat{\rho}|_\Sigma, \hat{Q}|_\Sigma)$ is an $(n-1)$ -dataset and its mass (in the sense of Definition 1.4) is equal to 0.*

Proof. Since $\mathcal{S} = \emptyset$, it follows that (Σ, \hat{g}) is a complete Riemannian manifold. Using Corollary 3.31, we can show that $(\Sigma, \hat{g}, \hat{\rho}|_\Sigma, \hat{Q}|_\Sigma)$ is an $(n-1)$ -dataset. Finally, it follows from (32) and (33) that $(\Sigma, \hat{g}, \hat{\rho}|_\Sigma, \hat{Q}|_\Sigma)$ has zero mass (in the sense of Definition 1.4). This completes the proof of Proposition 3.38.

In the remainder of this subsection, we assume that the singular set \mathcal{S} is non-empty. In this case, $n \geq 8$. Let Ψ denote the function defined in (31), and let ε_0 be chosen as in Corollary 3.36. We define a function $w : \Sigma \rightarrow \mathbb{R}$ by

$$(34) \quad w = 1 + \varepsilon_0 \Psi|_\Sigma.$$

Since Ψ is nonnegative, we have

$$(35) \quad w \geq 1$$

at each point on Σ . We define a conformal metric \tilde{g} on Σ by

$$(36) \quad \tilde{g} = w^{\frac{n+1}{n-3}} \hat{g}.$$

Moreover, we define a positive smooth function $\tilde{\rho}$ on Σ by

$$(37) \quad \tilde{\rho} = w^{-\frac{n+1}{2}} \hat{\rho}|_{\Sigma}.$$

Finally, we define a function \tilde{Q} on Σ by

$$(38) \quad \tilde{Q} = \frac{1}{2} w^{-\frac{n+1}{n-3}} \hat{Q}|_{\Sigma}.$$

Lemma 3.39. *We have*

$$w^{\frac{n+1}{n-3}} \tilde{Q} \leq \hat{Q} - \frac{n+1}{2(n-3)} w^{-1} \Delta_{\hat{g}} w - \frac{1}{2} w^{-1} \langle d \log \hat{\rho}, dw \rangle_{\hat{g}}$$

at each point on Σ .

Proof. Using (34), (35), and Corollary 3.37, we obtain

$$(39) \quad \Delta_{\Sigma} w + \frac{n-3}{n+1} \langle \nabla^{\Sigma} \log \hat{\rho}, \nabla^{\Sigma} w \rangle \leq \frac{n-3}{n+1} \hat{Q} \leq \frac{n-3}{n+1} \hat{Q} w$$

at each point on Σ . The assertion follows from (38) and (39). This completes the proof of Lemma 3.39.

Proposition 3.40. *Let f be a smooth test function on Σ with the property that f vanishes near the singular set and f is constant near infinity. Then*

$$\begin{aligned} & \int_{\Sigma} \tilde{\rho} |df|_{\tilde{g}}^2 d\text{vol}_{\tilde{g}} + \frac{1}{2} \int_{\Sigma} \tilde{\rho} \left(R_{\tilde{g}} - 2 \Delta_{\tilde{g}} \log \tilde{\rho} - \frac{n}{n+1} |d \log \tilde{\rho}|_{\tilde{g}}^2 \right) f^2 d\text{vol}_{\tilde{g}} \\ & \geq \int_{\Sigma} \tilde{\rho} \tilde{Q} f^2 d\text{vol}_{\tilde{g}} \end{aligned}$$

Proof. Note that

$$w^{\frac{n+1}{n-3}} |df|_{\tilde{g}}^2 = |df|_{\hat{g}}^2.$$

The standard formula for the change of the scalar curvature under a conformal change of the metric gives

$$\begin{aligned} w^{\frac{n+1}{n-3}} R_{\tilde{g}} &= R_{\hat{g}} - \frac{4(n-2)}{n-3} w^{-\frac{n+1}{4}} \Delta_{\hat{g}} (w^{\frac{n+1}{4}}) \\ &= R_{\hat{g}} - \frac{(n-2)(n+1)}{n-3} w^{-1} \Delta_{\hat{g}} w - \frac{(n-2)(n+1)}{4} w^{-2} |dw|_{\hat{g}}^2. \end{aligned}$$

Moreover,

$$\begin{aligned}
w^{\frac{n+1}{n-3}} \Delta_{\tilde{g}} \log \tilde{\rho} &= w^{-\frac{n+1}{2}} \operatorname{div}_{\tilde{g}}(w^{\frac{n+1}{2}} d \log \tilde{\rho}) \\
&= \Delta_{\tilde{g}} \log \tilde{\rho} + \frac{n+1}{2} \langle d \log \tilde{\rho}, d \log w \rangle \\
&= \Delta_{\tilde{g}} \log \hat{\rho} - \frac{n+1}{2} \Delta_{\tilde{g}} \log w + \frac{n+1}{2} \langle d \log \hat{\rho}, d \log w \rangle_{\tilde{g}} \\
&\quad - \frac{(n+1)^2}{4} |d \log w|_{\tilde{g}}^2 \\
&= \Delta_{\tilde{g}} \log \hat{\rho} - \frac{n+1}{2} w^{-1} \Delta_{\tilde{g}} w + \frac{n+1}{2} w^{-1} \langle d \log \hat{\rho}, dw \rangle_{\tilde{g}} \\
&\quad - \frac{(n-1)(n+1)}{4} w^{-2} |dw|_{\tilde{g}}^2
\end{aligned}$$

and

$$\begin{aligned}
w^{\frac{n+1}{n-3}} |d \log \tilde{\rho}|_{\tilde{g}}^2 &= |d \log \tilde{\rho}|_{\tilde{g}}^2 \\
&= \left| d \log \hat{\rho} - \frac{n+1}{2} d \log w \right|_{\tilde{g}}^2 \\
&= |d \log \hat{\rho}|_{\tilde{g}}^2 - (n+1) w^{-1} \langle d \log \hat{\rho}, dw \rangle_{\tilde{g}} \\
&\quad + \frac{(n+1)^2}{4} w^{-2} |dw|_{\tilde{g}}^2.
\end{aligned}$$

Using these identities together with Lemma 3.39, we obtain the pointwise inequality

$$\begin{aligned}
&w^{\frac{n+1}{n-3}} |df|_{\tilde{g}}^2 + \frac{1}{2} w^{\frac{n+1}{n-3}} \left(R_{\tilde{g}} - 2 \Delta_{\tilde{g}} \log \tilde{\rho} - \frac{n}{n+1} |d \log \tilde{\rho}|_{\tilde{g}}^2 \right) f^2 - w^{\frac{n+1}{n-3}} \tilde{Q} f^2 \\
&\geq |df|_{\tilde{g}}^2 + \frac{1}{2} \left(R_{\tilde{g}} - 2 \Delta_{\tilde{g}} \log \hat{\rho} - \frac{n}{n+1} |d \log \hat{\rho}|_{\tilde{g}}^2 \right) f^2 - \hat{Q} f^2.
\end{aligned}$$

This finally implies

$$\begin{aligned}
&\int_{\Sigma} \tilde{\rho} |df|_{\tilde{g}}^2 d \operatorname{vol}_{\tilde{g}} + \frac{1}{2} \int_{\Sigma} \tilde{\rho} \left(R_{\tilde{g}} - 2 \Delta_{\tilde{g}} \log \tilde{\rho} - \frac{n}{n+1} |d \log \tilde{\rho}|_{\tilde{g}}^2 \right) f^2 d \operatorname{vol}_{\tilde{g}} \\
&\quad - \int_{\Sigma} \tilde{\rho} \tilde{Q} f^2 d \operatorname{vol}_{\tilde{g}} \\
&\geq \int_{\Sigma} \hat{\rho} |df|_{\tilde{g}}^2 d \operatorname{vol}_{\tilde{g}} + \frac{1}{2} \int_{\Sigma} \hat{\rho} \left(R_{\tilde{g}} - 2 \Delta_{\tilde{g}} \log \hat{\rho} - \frac{n}{n+1} |d \log \hat{\rho}|_{\tilde{g}}^2 \right) f^2 d \operatorname{vol}_{\tilde{g}} \\
&\quad - \int_{\Sigma} \hat{\rho} \hat{Q} f^2 d \operatorname{vol}_{\tilde{g}}.
\end{aligned}$$

The expression on the right hand side is nonnegative by Corollary 3.31. This completes the proof of Proposition 3.40.

Proposition 3.41. *The metric \tilde{g} on Σ is complete.*

Proof. By Proposition 3.34, we can find small positive constants c_1 and c_2 such that

$$w(x) \geq c_2 d_{(M,g)}(x, \mathcal{S})^{-2}$$

for each point $x \in \Sigma$ satisfying $d_{(M,g)}(x, \mathcal{S}) \leq c_1$.

Suppose now that y_0 and y_1 are two points in Σ , and suppose that $\sigma : [0, 1] \rightarrow \Sigma$ is a smooth path satisfying $\sigma(0) = y_0$ and $\sigma(1) = y_1$. Then

$$\frac{d}{dt} d_{(M,g)}(\sigma(t), \mathcal{S}) \geq -|\sigma'(t)|_{\hat{g}}$$

whenever $d_{(M,g)}(\sigma(t), \mathcal{S}) \leq c_1$. Here, the derivative is understood in the sense of liminf of backward difference quotients. This implies

$$\begin{aligned} \frac{d}{dt} (d_{(M,g)}(\sigma(t), \mathcal{S})^{-\frac{4}{n-3}}) &\leq \frac{4}{n-3} d_{(M,g)}(\sigma(t), \mathcal{S})^{-\frac{n+1}{n-3}} |\sigma'(t)|_{\hat{g}} \\ &\leq \frac{4}{n-3} c_2^{-\frac{n+1}{2(n-3)}} w(\sigma(t))^{\frac{n+1}{2(n-3)}} |\sigma'(t)|_{\hat{g}} \\ &= \frac{4}{n-3} c_2^{-\frac{n+1}{2(n-3)}} |\sigma'(t)|_{\hat{g}} \end{aligned}$$

whenever $d_{(M,g)}(\sigma(t), \mathcal{S}) \leq c_1$. Here, the derivative is understood in the sense of limsup of backward difference quotients. From this, we deduce that

$$\begin{aligned} d_{(M,g)}(y_1, \mathcal{S})^{-\frac{4}{n-3}} &\leq \max \left\{ d_{(M,g)}(y_0, \mathcal{S})^{-\frac{4}{n-3}}, c_1^{-\frac{4}{n-3}} \right\} \\ &\quad + \frac{4}{n-3} c_2^{-\frac{n+1}{2(n-3)}} \int_0^1 |\sigma'(t)|_{\hat{g}} dt. \end{aligned}$$

Thus,

$$\begin{aligned} d_{(M,g)}(y_1, \mathcal{S})^{-\frac{4}{n-3}} &\leq \max \left\{ d_{(M,g)}(y_0, \mathcal{S})^{-\frac{4}{n-3}}, c_1^{-\frac{4}{n-3}} \right\} \\ &\quad + \frac{4}{n-3} c_2^{-\frac{n+1}{2(n-3)}} d_{(\Sigma, \tilde{g})}(y_0, y_1), \end{aligned}$$

where $d_{(\Sigma, \tilde{g})}(y_0, y_1)$ denotes the Riemannian distance of y_0 and y_1 with respect to the metric \tilde{g} on Σ . Therefore, the metric \tilde{g} on Σ is complete. This completes the proof of Proposition 3.41.

Corollary 3.42. *If $\mathcal{S} \neq \emptyset$, then $(\Sigma, \tilde{g}, \tilde{\rho}, \tilde{Q})$ is an $(n-1)$ -dataset and its mass (in the sense of Definition 1.4) is equal to 0.*

Proof. It follows from Proposition 3.41 that (Σ, \tilde{g}) is a complete Riemannian manifold. Using Proposition 3.40, we can show that $(\Sigma, \tilde{g}, \tilde{\rho}, \tilde{Q})$ is an $(n-1)$ -dataset. We next observe that the function Ψ vanishes near infinity. Consequently, $w = 1$ near infinity. Using (32) and (33), we conclude that

$$\tilde{g} = \hat{g} = dx_1 \otimes dx_1 + \dots + dx_{n-1} \otimes dx_{n-1} + O((x_1^2 + \dots + x_{n-1}^2)^{\frac{2-n}{2}})$$

and

$$\tilde{\rho} = \hat{\rho}|_{\Sigma} = 1 + O((x_1^2 + \dots + x_{n-1}^2)^{\frac{2-n}{2}})$$

near infinity. Thus, $(\Sigma, \tilde{g}, \tilde{\rho}, \tilde{Q})$ has zero mass (in the sense of Definition 1.4). This completes the proof of Corollary 3.42.

Combining Proposition 3.38 and Corollary 3.42, we conclude that Theorem 1.5 is false in dimension $n-1$. This contradicts the inductive hypothesis. The proof of Theorem 1.5 is now complete.

APPENDIX A. A DIVERGENCE IDENTITY

In this section, we state a divergence identity that generalizes Proposition A.2 in [5] (see also [3]).

Proposition A.1. *Let (M, g) be a Riemannian manifold and let $\hat{\rho}$ be a positive function on M . Let Σ be a two-sided hypersurface in M satisfying $H_\Sigma + \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle = \Phi$, where Φ is a smooth function on M . Let V be a smooth vector field on M , and let $W = D_V V$. We define a function f on Σ by $f = \langle V, \nu_\Sigma \rangle$. Moreover, we define a tangential vector field Z along Σ by*

$$Z = D_{V^{\text{tan}}}^\Sigma(V^{\text{tan}}) - \text{div}_\Sigma(V^{\text{tan}}) V^{\text{tan}} + 2 \sum_{k=1}^{n-1} h_\Sigma(V^{\text{tan}}, e_k) \langle V, \nu_\Sigma \rangle e_k.$$

Then

$$\begin{aligned} & \hat{\rho} |\nabla^\Sigma f|^2 - \hat{\rho} (\text{Ric}(\nu_\Sigma, \nu_\Sigma) + |h_\Sigma|^2) f^2 + (D^2 \hat{\rho})(\nu_\Sigma, \nu_\Sigma) f^2 - \hat{\rho}^{-1} \langle \nabla \hat{\rho}, \nu_\Sigma \rangle^2 f^2 \\ & + \text{div}_\Sigma(\hat{\rho} W^{\text{tan}}) - \text{div}_\Sigma(\hat{\rho} Z) + \text{div}_\Sigma(\langle V^{\text{tan}}, \nabla^\Sigma \hat{\rho} \rangle V^{\text{tan}}) \\ & - \hat{\rho} \langle \nabla \Phi, \nu_\Sigma \rangle f^2 + \text{div}(\hat{\rho} \Phi V) \langle V, \nu_\Sigma \rangle + \text{div}_\Sigma(\hat{\rho} \Phi \langle V, \nu_\Sigma \rangle V^{\text{tan}}) \\ & = \frac{1}{2} \hat{\rho} \sum_{k=1}^{n-1} (\mathcal{L}_V \mathcal{L}_V g)(e_k, e_k) + V(V(\hat{\rho})) \\ & - \frac{1}{2} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_l) (\mathcal{L}_V g)(e_k, e_l) \\ & + \frac{1}{4} \hat{\rho} \sum_{k,l=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) (\mathcal{L}_V g)(e_l, e_l) \\ & + V(\hat{\rho}) \sum_{k=1}^{n-1} (\mathcal{L}_V g)(e_k, e_k) \end{aligned}$$

at each point on Σ . Here, $\{e_1, \dots, e_{n-1}\}$ denotes a local orthonormal frame on Σ .

Proof. We adapt the proof of Proposition A.2 in [5]. As in [5], we write $Z = Z^{(1)} + Z^{(2)}$, where

$$Z^{(1)} = D_{V^{\text{tan}}}^\Sigma(V^{\text{tan}}) - \text{div}_\Sigma(V^{\text{tan}}) V^{\text{tan}}$$

and

$$Z^{(2)} = 2 \sum_{k=1}^{n-1} h_{\Sigma}(V^{\tan}, e_k) \langle V, \nu_{\Sigma} \rangle e_k.$$

Using the Gauss equations and the identity $\langle D_{e_k} V, e_l \rangle = \langle D_{e_k} V^{\tan}, e_l \rangle + h_{\Sigma}(e_k, e_l) \langle V, \nu_{\Sigma} \rangle$ for $k, l \in \{1, \dots, n-1\}$, we compute

$$\begin{aligned} & \operatorname{div}_{\Sigma}(Z^{(1)}) \\ &= \sum_{k,l=1}^{n-1} \langle D_{e_k} V^{\tan}, e_l \rangle \langle D_{e_l} V^{\tan}, e_k \rangle - \sum_{k,l=1}^{n-1} \langle D_{e_k} V^{\tan}, e_k \rangle \langle D_{e_l} V^{\tan}, e_l \rangle \\ &+ \operatorname{Ric}_{\Sigma}(V^{\tan}, V^{\tan}) \\ &= \sum_{k,l=1}^{n-1} \langle D_{e_k} V, e_l \rangle \langle D_{e_l} V, e_k \rangle - \sum_{k,l=1}^{n-1} \langle D_{e_k} V, e_k \rangle \langle D_{e_l} V, e_l \rangle \\ &- 2 \sum_{k,l=1}^{n-1} h_{\Sigma}(e_k, e_l) \langle D_{e_k} V^{\tan}, e_l \rangle \langle V, \nu_{\Sigma} \rangle + 2 H_{\Sigma} \sum_{k=1}^{n-1} \langle D_{e_k} V, e_k \rangle \langle V, \nu_{\Sigma} \rangle \\ &- H_{\Sigma}^2 \langle V, \nu_{\Sigma} \rangle^2 - |h_{\Sigma}|^2 \langle V, \nu_{\Sigma} \rangle^2 + H_{\Sigma} h_{\Sigma}(V^{\tan}, V^{\tan}) - h_{\Sigma}^2(V^{\tan}, V^{\tan}) \\ &+ \sum_{k=1}^{n-1} R(V^{\tan}, e_k, V^{\tan}, e_k). \end{aligned}$$

Using the Codazzi equations, we obtain

$$\begin{aligned} \operatorname{div}_{\Sigma}(Z^{(2)}) &= 2 \sum_{k,l=1}^{n-1} h_{\Sigma}(e_k, e_l) \langle D_{e_k} V^{\tan}, e_l \rangle \langle V, \nu_{\Sigma} \rangle \\ &+ 2 \sum_{k=1}^{n-1} h_{\Sigma}(V^{\tan}, e_k) \langle D_{e_k} V, \nu_{\Sigma} \rangle + 2 h_{\Sigma}^2(V^{\tan}, V^{\tan}) \\ &+ 2 \sum_{k=1}^{n-1} R(V^{\tan}, e_k, \nu_{\Sigma}, e_k) \langle V, \nu_{\Sigma} \rangle + 2 \langle \nabla^{\Sigma} H_{\Sigma}, V^{\tan} \rangle \langle V, \nu_{\Sigma} \rangle. \end{aligned}$$

Moreover,

$$|\nabla^{\Sigma} f|^2 = \sum_{k=1}^{n-1} \langle D_{e_k} V, \nu_{\Sigma} \rangle^2 + 2 \sum_{k=1}^{n-1} h_{\Sigma}(V^{\tan}, e_k) \langle D_{e_k} V, \nu_{\Sigma} \rangle + h_{\Sigma}^2(V^{\tan}, V^{\tan}).$$

Putting these facts together, we obtain

$$\begin{aligned}
& \operatorname{div}_\Sigma Z - |\nabla^\Sigma f|^2 + (\operatorname{Ric}(\nu_\Sigma, \nu_\Sigma) + |h_\Sigma|^2) f^2 \\
&= \sum_{k,l=1}^{n-1} \langle D_{e_k} V, e_l \rangle \langle D_{e_l} V, e_k \rangle - \sum_{k,l=1}^{n-1} \langle D_{e_k} V, e_k \rangle \langle D_{e_l} V, e_l \rangle \\
&\quad - \sum_{k=1}^{n-1} \langle D_{e_k} V, \nu_\Sigma \rangle^2 + \sum_{k=1}^{n-1} R(V, e_k, V, e_k) \\
&\quad + H_\Sigma h_\Sigma(V^{\tan}, V^{\tan}) - H_\Sigma^2 \langle V, \nu_\Sigma \rangle^2 \\
&\quad + 2 H_\Sigma \sum_{k=1}^{n-1} \langle D_{e_k} V, e_k \rangle \langle V, \nu_\Sigma \rangle + 2 \langle \nabla^\Sigma H_\Sigma, V^{\tan} \rangle \langle V, \nu_\Sigma \rangle.
\end{aligned}$$

Using the identity $H_\Sigma + \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle = \Phi$, it follows that

$$\begin{aligned}
& \operatorname{div}_\Sigma(\hat{\rho} Z) - \operatorname{div}_\Sigma(\langle V^{\tan}, \nabla^\Sigma \hat{\rho} \rangle V^{\tan}) \\
&\quad - \hat{\rho} |\nabla^\Sigma f|^2 + \hat{\rho} (\operatorname{Ric}(\nu_\Sigma, \nu_\Sigma) + |h_\Sigma|^2) f^2 \\
&\quad - (D^2 \hat{\rho})(\nu_\Sigma, \nu_\Sigma) f^2 + \hat{\rho}^{-1} \langle \nabla \hat{\rho}, \nu_\Sigma \rangle^2 f^2 \\
&\quad - 2 \hat{\rho} \langle \nabla^\Sigma \Phi, V^{\tan} \rangle \langle V, \nu_\Sigma \rangle - 2 \hat{\rho} \Phi \operatorname{div}_\Sigma(V^{\tan}) \langle V, \nu_\Sigma \rangle \\
&\quad - 2 \Phi \langle \nabla^\Sigma \hat{\rho}, V^{\tan} \rangle \langle V, \nu_\Sigma \rangle - \hat{\rho} \Phi h_\Sigma(V^{\tan}, V^{\tan}) - \hat{\rho} \Phi^2 \langle V, \nu_\Sigma \rangle^2 \\
&= \hat{\rho} \sum_{k,l=1}^{n-1} \langle D_{e_k} V, e_l \rangle \langle D_{e_l} V, e_k \rangle - \hat{\rho} \sum_{k,l=1}^{n-1} \langle D_{e_k} V, e_k \rangle \langle D_{e_l} V, e_l \rangle \\
&\quad - \hat{\rho} \sum_{k=1}^{n-1} \langle D_{e_k} V, \nu_\Sigma \rangle^2 + \hat{\rho} \sum_{k=1}^{n-1} R(V, e_k, V, e_k) \\
&\quad - 2 V(\hat{\rho}) \sum_{k=1}^{n-1} \langle D_{e_k} V, e_k \rangle - (D^2 \hat{\rho})(V, V).
\end{aligned}$$

We next observe that

$$(\mathcal{L}_V \mathcal{L}_V g)(X, Y) - (\mathcal{L}_W g)(X, Y) = 2 \langle D_X V, D_Y V \rangle - 2 R(V, X, V, Y)$$

for all vector fields X, Y on M . Moreover,

$$V(V(\hat{\rho})) - W(\hat{\rho}) = (D^2 \hat{\rho})(V, V).$$

Using these identities together with the identity $H_\Sigma + \langle \nabla \log \hat{\rho}, \nu_\Sigma \rangle = \Phi$, we obtain

$$\begin{aligned} & \frac{1}{2} \hat{\rho} \sum_{k=1}^{n-1} (\mathcal{L}_V \mathcal{L}_V g)(e_k, e_k) + V(V(\hat{\rho})) - \operatorname{div}_\Sigma(\hat{\rho} W^{\tan}) - \hat{\rho} \Phi \langle W, \nu_\Sigma \rangle \\ &= \hat{\rho} \sum_{k=1}^{n-1} |D_{e_k} V|^2 - \hat{\rho} \sum_{k=1}^{n-1} R(V, e_k, V, e_k) + (D^2 \hat{\rho})(V, V) \\ &= \hat{\rho} \sum_{k,l=1}^{n-1} \langle D_{e_k} V, e_l \rangle^2 + \hat{\rho} \sum_{k=1}^{n-1} \langle D_{e_k} V, \nu_\Sigma \rangle^2 - \hat{\rho} \sum_{k=1}^{n-1} R(V, e_k, V, e_k) + (D^2 \hat{\rho})(V, V). \end{aligned}$$

Putting these facts together, we conclude that

$$\begin{aligned} & \frac{1}{2} \hat{\rho} \sum_{k=1}^{n-1} (\mathcal{L}_V \mathcal{L}_V g)(e_k, e_k) + V(V(\hat{\rho})) - \operatorname{div}_\Sigma(\hat{\rho} W^{\tan}) \\ &+ \operatorname{div}_\Sigma(\hat{\rho} Z) - \operatorname{div}_\Sigma(\langle V^{\tan}, \nabla^\Sigma \hat{\rho} \rangle V^{\tan}) \\ &- \hat{\rho} |\nabla^\Sigma f|^2 + \hat{\rho} (\operatorname{Ric}(\nu_\Sigma, \nu_\Sigma) + |h_\Sigma|^2) f^2 \\ &- (D^2 \hat{\rho})(\nu_\Sigma, \nu_\Sigma) f^2 + \hat{\rho}^{-1} \langle \nabla \hat{\rho}, \nu_\Sigma \rangle^2 f^2 \\ &- \hat{\rho} \Phi \langle W, \nu_\Sigma \rangle - 2\hat{\rho} \langle \nabla^\Sigma \Phi, V^{\tan} \rangle \langle V, \nu_\Sigma \rangle - 2\hat{\rho} \Phi \operatorname{div}_\Sigma(V^{\tan}) \langle V, \nu_\Sigma \rangle \\ &- 2\Phi \langle \nabla^\Sigma \hat{\rho}, V^{\tan} \rangle \langle V, \nu_\Sigma \rangle - \hat{\rho} \Phi h_\Sigma(V^{\tan}, V^{\tan}) - \hat{\rho} \Phi^2 \langle V, \nu_\Sigma \rangle^2 \\ &= \hat{\rho} \sum_{k,l=1}^{n-1} \langle D_{e_k} V, e_l \rangle^2 + \hat{\rho} \sum_{k,l=1}^{n-1} \langle D_{e_k} V, e_l \rangle \langle D_{e_l} V, e_k \rangle \\ &- \hat{\rho} \sum_{k,l=1}^{n-1} \langle D_{e_k} V, e_k \rangle \langle D_{e_l} V, e_l \rangle - 2V(\hat{\rho}) \sum_{k=1}^{n-1} \langle D_{e_k} V, e_k \rangle. \end{aligned}$$

Finally, a straightforward calculation gives

$$\begin{aligned} & \hat{\rho} \Phi \langle W, \nu_\Sigma \rangle + 2\hat{\rho} \langle \nabla^\Sigma \Phi, V^{\tan} \rangle \langle V, \nu_\Sigma \rangle + 2\hat{\rho} \Phi \operatorname{div}_\Sigma(V^{\tan}) \langle V, \nu_\Sigma \rangle \\ &+ 2\Phi \langle \nabla^\Sigma \hat{\rho}, V^{\tan} \rangle \langle V, \nu_\Sigma \rangle + \hat{\rho} \Phi h_\Sigma(V^{\tan}, V^{\tan}) + \hat{\rho} \Phi^2 \langle V, \nu_\Sigma \rangle^2 \\ &= -\hat{\rho} \langle \nabla \Phi, \nu_\Sigma \rangle \langle V, \nu_\Sigma \rangle^2 + \operatorname{div}(\hat{\rho} \Phi V) \langle V, \nu_\Sigma \rangle + \operatorname{div}_\Sigma(\hat{\rho} \Phi \langle V, \nu_\Sigma \rangle V^{\tan}). \end{aligned}$$

From this, the assertion follows easily. This completes the proof of Proposition A.1.

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