ON THE SPACETIME POSITIVE MASS THEOREM IN ARBITRARY DIMENSION

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Abstract. We describe how the spacetime positive energy theorem in dimension $n \geq 4$ follows from our recent work on the Riemannian version of the positive energy theorem. Our work builds on the fundamental work of Schoen and Yau in dimension $n = 3$ and its extension to dimension $3 \leq n \leq 7$ in the remarkable work of Eichmair. The proof relies on the Jang equation with a capillary term. We also use the shielding principle from the work of Lesourd-Unger-Yau.

In combination with well-known results in the literature, our work implies the spacetime positive mass theorem in dimension $n \geq 4$.

1. Introduction

Let (M, g) be a Riemannian manifold of dimension $n \geq 4$, and let q be a symmetric $(0, 2)$-tensor on M. We define a scalar function μ on M by

$$\mu = \frac{1}{2} \, (R_g - |q|_g^2 + \operatorname{tr}_g(q)^2).$$

We define a one-form J on M by

$$J_k = g^{ij} \, (D_i q_{jk} - D_k q_{ij}) = g^{ij} \, D_i q_{jk} - \partial_k \operatorname{tr}_g(q).$$

Theorem 1.1. *Suppose that (M, g) is a Riemannian manifold of dimension $n \geq 4$, and q is a symmetric $(0, 2)$-tensor on M. We assume that there exists a compact subset $K \subset M$ such that $M \backslash K$ is diffeomorphic to the complement of a ball in $\mathbb{R}^n$. Moreover, we assume that there exist real numbers α and $\delta > 0$ such that*

$$|\bar{D}^m (g - (1 + \alpha \, r^{2-n}) \, \bar{g})|_{\bar{g}} \leq C(m) \, r^{2-n-m-2\delta}$$

and

$$|\bar{D}^m q|_{\bar{g}} \leq C(m) \, r^{1-n-m}$$

for all points near infinity and for every nonnegative integer m. Here, $\bar{g}$ denotes the Euclidean metric, $\bar{D}^m$ denotes the covariant derivative of order m with respect to $\bar{g}$, and $r = \sqrt{x_1^2 + \ldots + x_n^2}$ denotes the radial coordinate near infinity. Finally, we assume that $\mu - |J|_g > 0$ at each point in M. Then $\alpha > 0$.

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Remark 1.2 (The spacetime positive mass theorem and the rigidity statement)**.** In view of the density theorem of Eichmair-Huang-Lee-Schoen (see [8], Theorem 18), Theorem 1.1 implies the usual spacetime positive energy theorem for initial data sets satisfying the dominant energy condition $\mu - |J|_g \geq 0$. Moreover, the density theorem of Eichmair-Huang-Lee-Schoen makes it possible to weaken the asymptotic assumptions. We refer to Definition 3 in [8] for the relevant definitions and asymptotic assumptions. Note that the tensor q in our paper is denoted by k in [8].

Furthermore, as explained in the final remark on p. 119 in [8], one may use the boost argument of Christodoulou and O'Murchadha (see [3], Theorem 6.1) to deduce the spacetime positive mass theorem from the spacetime positive energy theorem. In light of that, Theorem 1.1 implies the spacetime positive mass theorem in dimension $n \geq 4$.

Finally, in the case of equality, (M, g, q) arises as a spacelike hypersurface in the Minkowski spacetime. This follows directly from work of Huang and Lee (see [11], Theorem 3).

Remark 1.3 (The hyperbolic positive energy theorem)**.** The positive energy theorem for asymptotically hyperbolic manifolds follows from the spacetime positive energy theorem for asymptotically flat initial data sets satisfying the dominant energy condition. This was shown in a remarkable work of Chruściel and Delay [4]. In combination with the work of Chruściel-Delay [4], Theorem 1.1 implies the hyperbolic positive energy theorem in dimension $n \geq 4$.

The spacetime positive energy theorem in dimension 3 was proved in a famous work of Schoen and Yau [19]. In their proof, Schoen and Yau used the Jang equation (see [12]) to reduce the spacetime positive energy theorem to the Riemannian positive mass theorem [17],[18]. The spacetime version of the positive mass theorem in dimension $3 \leq n \leq 7$ was proved by Eichmair [7] using the Jang equation (see also Remark 1.2 above), and by Eichmair-Huang-Lee-Schoen [8] using marginally outer trapped surfaces. The Jang equation was studied further in [5], [6], and [9]. Finally, Lohkamp [16] has proposed a proof of the spacetime positive mass theorem in all dimensions.

Our proof of Theorem 1.1 relies on a modified Jang equation that includes a capillary term. This modified Jang equation is similar to the one introduced by Schoen and Yau [19]. The latter equation also plays an important role in Eichmair's work [7].

2. Proof of Theorem 1.1

Suppose that (M, g, q) is a triplet satisfying the assumptions of Theorem 1.1. We claim that $\alpha > 0$. To prove this, we argue by contradiction. Suppose that $\alpha \leq 0$.

2.1. **A barrier construction from the work of Schoen-Yau and Eichmair.** In this subsection, we recall a barrier construction for the Jang equation. This construction goes back to the fundamental work of Schoen and Yau [19] in the three-dimensional case, and was later extended by Eichmair [7] to the higher-dimensional case. In this paper, we focus on the case $n \geq 4$ treated in Eichmair's work [7].

Proposition 2.1 (R. Schoen, S.T. Yau [19]; M. Eichmair [7])**.** *If r_0 is sufficiently large, then the following holds. We define a positive function $b : (r_0, \infty) \to (0, \infty)$ by*

$$b(s) = r_0 \int_{r_0^{-1} s}^{\infty} (t^{2n-4} - 1)^{-\frac{1}{2}} \, dt$$

for $s \in (r_0, \infty)$. Moreover, we define a positive function Υ on the domain $\{r > r_0\}$ by $\Upsilon = b \circ r$, where r denotes the radial coordinate near infinity. Then

$$g^{ik} \, g^{jl} \, \big(g_{ij} - (1 + |d\Upsilon|_g^2)^{-1} \, \partial_i \Upsilon \, \partial_j \Upsilon\big) \, \big((1 + |d\Upsilon|_g^2)^{-\frac{1}{2}} \, D^2_{k,l} \Upsilon - \lambda \, q_{kl}\big) < 0$$

at each point in the domain $\{r > r_0\}$ and for each $\lambda \in [-1, 1]$.

Proof. The function b satisfies the ODE

$$s \, b''(s) + (n-1) \, (1 + b'(s)^2) \, b'(s) = -(1 + b'(s)^2)^{\frac{3}{2}} \, r_0^{n-2} \, s^{2-n} \tag{1}$$

for $s \in (r_0, \infty)$. The function $\Upsilon = b \circ r$ satisfies

$$\begin{aligned}
&g^{ik} \, g^{jl} \, \big(g_{ij} - (1 + |d\Upsilon|_g^2)^{-1} \, \partial_i \Upsilon \, \partial_j \Upsilon\big) \, (1 + |d\Upsilon|_g^2)^{-\frac{1}{2}} \, D^2_{k,l} \Upsilon \\
&= (1 + b'(r)^2 \, |dr|_g^2)^{-\frac{3}{2}} \, b''(r) \, |dr|_g^2 + (n-1) \, (1 + b'(r)^2 \, |dr|_g^2)^{-\frac{1}{2}} \, r^{-1} \, b'(r) \, |dr|_g^2 \\
&+ (1 + b'(r)^2 \, |dr|_g^2)^{-\frac{1}{2}} \, r^{-1} \, b'(r) \, \big(g_{ij} - (1 + b'(r)^2 \, |dr|_g^2)^{-1} \, b'(r)^2 \, \partial_i r \, \partial_j r\big) \, T^{ij},
\end{aligned}$$

where

$$T^{ij} = g^{ik} \, g^{jl} \, (r \, D^2_{k,l} r + \partial_k r \, \partial_l r - |dr|_g^2 \, g_{kl}).$$

Using (1), we obtain

$$\begin{aligned}
&g^{ik} \, g^{jl} \, \big(g_{ij} - (1 + |d\Upsilon|_g^2)^{-1} \, \partial_i \Upsilon \, \partial_j \Upsilon\big) \, (1 + |d\Upsilon|_g^2)^{-\frac{1}{2}} \, D^2_{k,l} \Upsilon \\
&= -(1 + b'(r)^2 \, |dr|_g^2)^{-\frac{3}{2}} \, (1 + b'(r)^2)^{\frac{3}{2}} \, r_0^{n-2} \, r^{1-n} \, |dr|_g^2 \\
&+ (n-1) \, (1 + b'(r)^2 \, |dr|_g^2)^{-\frac{3}{2}} \, r^{-1} \, b'(r)^3 \, |dr|_g^2 \, (|dr|_g^2 - 1) \\
&+ (1 + b'(r)^2 \, |dr|_g^2)^{-\frac{1}{2}} \, r^{-1} \, b'(r) \, \big(g_{ij} - b'(r)^2 \, (1 + b'(r)^2 \, |dr|_g^2)^{-1} \, \partial_i r \, \partial_j r\big) \, T^{ij}.
\end{aligned}$$

Using the inequalities

$$(1 + b'(r)^2 \, |dr|_g^2)^{-\frac{3}{2}} \, (1 + b'(r)^2)^{\frac{3}{2}} \geq \min\{1, |dr|_g^{-3}\},$$

$$(1 + b'(r)^2 \, |dr|_g^2)^{-1} \, b'(r)^2 \, |dr|_g^2 \leq 1,$$

and

$$\big|\big(g_{ij} - b'(r)^2 \, (1 + b'(r)^2 \, |dr|_g^2)^{-1} \, \partial_i r \, \partial_j r\big) \, T^{ij}\big| \leq n \, |T|_g,$$

we conclude that

$$\begin{aligned} &g^{ik}\, g^{jl}\, \big(g_{ij} - (1 + |d\Upsilon|_g^2)^{-1}\, \partial_i \Upsilon\, \partial_j \Upsilon\big)\, (1 + |d\Upsilon|_g^2)^{-\frac{1}{2}}\, D^2_{k,l} \Upsilon \\ &\leq -r_0^{n-2}\, r^{1-n}\, \min\{|dr|_g^2, |dr|_g^{-1}\} \\ &+ (n-1)\, r^{-1}\, |dr|_g^{-1}\, \big||dr|_g^2 - 1\big| + r^{-1}\, |dr|_g^{-1}\, n\, |T|_g. \end{aligned}$$

By assumption, $\big||dr|_g^2 - 1\big| \leq C\, r^{2-n}$ and $|T|_g \leq C\, r^{2-n}$. This implies

$$\begin{aligned} &g^{ik}\, g^{jl}\, \big(g_{ij} - (1 + |d\Upsilon|_g^2)^{-1}\, \partial_i \Upsilon\, \partial_j \Upsilon\big)\, (1 + |d\Upsilon|_g^2)^{-\frac{1}{2}}\, D^2_{k,l} \Upsilon \\ &\leq -r_0^{n-2}\, r^{1-n}\, \min\{|dr|_g^2, |dr|_g^{-1}\} + C\, r^{1-n}. \end{aligned}$$

Finally,

$$\big|g^{ik}\, g^{jl}\, \big(g_{ij} - (1 + |d\Upsilon|_g^2)^{-1}\, \partial_i \Upsilon\, \partial_j \Upsilon\big)\, q_{kl}\big| \leq n\, |q|_g \leq C\, r^{1-n}.$$

Putting these facts together, the assertion follows. This completes the proof of Proposition 2.1.

In the following, we fix r_0 so that the conclusion of Proposition 2.1 holds.

Proposition 2.2. *We have $b(s) \leq 2\, r_0^{n-2}\, s^{3-n}$ for $s \in (2r_0, \infty)$.*

Proof. Note that $t^{2n-4} - 1 \geq (1 - 2^{4-2n})\, t^{2n-4}$ for $t \in (2, \infty)$. This implies

$$\begin{aligned} b(s) &= r_0 \int_{r_0^{-1} s}^{\infty} (t^{2n-4} - 1)^{-\frac{1}{2}}\, dt \\ &\leq (1 - 2^{4-2n})^{-\frac{1}{2}}\, r_0 \int_{r_0^{-1} s}^{\infty} t^{2-n}\, dt \\ &= (n-3)^{-1}\, (1 - 2^{4-2n})^{-\frac{1}{2}}\, r_0^{n-2}\, s^{3-n} \end{aligned}$$

for $s \in (2r_0, \infty)$. Finally, $(n-3)^{-1}\, (1 - 2^{4-2n})^{-\frac{1}{2}} \leq 2$ since $n \geq 4$. This completes the proof of Proposition 2.2.

2.2. **The choice of κ_0, κ_1, s_0, s_1, and τ.** In this subsection, we fix various parameters that will be needed in the subsequent arguments. Let r_0 be chosen as in Proposition 2.1. We fix a smooth function $\zeta : M \to [0, 1]$ with the property that $\zeta = 0$ on $\{r > 8r_0\}$ and $\zeta = 1$ on $M \setminus \{r > 4r_0\}$. By assumption, $\mu - |J|_g > 0$ at each point in M. Since the function ζ has compact support, we can find positive constants κ_0 and κ_1 such that

$$\mu - |J|_g - \kappa_0^2\, |d\zeta|_g^2 - \kappa_1\, \zeta^2\, n\, |q|_g > 0$$

at each point in M. Consequently, we can find a positive smooth function Q on M such that

$$\mu - |J|_g - \kappa_0^2\, |d\zeta|_g^2 - \kappa_1\, \zeta^2\, n\, |q|_g \geq Q \tag{2}$$

at each point in M and

$$|\bar{D}^m Q|_{\bar{g}} \leq C(m)\, r^{-n-m-2\delta} \tag{3}$$

near infinity for every nonnegative integer m.

We next define $E_0 = \{r > 8r_0\}$. Since the function Q is strictly positive at each point in M, we can find positive real numbers s_0 and s_1 such that $s_1 \geq s_0$ and

$$Q(x) > \frac{128}{s_1 s_0} \tag{4}$$

for each point $x \in \mathcal{N}_g(E_0, 2s_0) \setminus E_0$. Having chosen s_0 and s_1 in this way, we fix $\tau > 0$ sufficiently small so that

$$2^{10-3n}\, r_0 + s_1 + 2s_0 \leq \frac{1}{2}\, \min\{\kappa_0\, \tau^{-1}, \kappa_1\, \tau^{-2}\}. \tag{5}$$

2.3. **A modified Jang equation.** In this subsection, we construct a solution of a modified Jang equation. To that end, we solve a sequence of Dirichlet problems on bounded domains and pass to the limit. Alternatively, one could prove existence via the Perron method as in Eichmair's work [5].

Let us consider a sequence of real numbers r_j such that $r_j > 32r_0$ for each j and $r_j \to \infty$ as $j \to \infty$. For each j, we define a compact domain $M^{(j)} \subset M$ by $M^{(j)} = M \setminus \{r > r_j\}$. We consider the following boundary value problem on $M^{(j)}$, which is inspired by work of Jang [12], Schoen-Yau [19], and Eichmair [7].

Definition 2.3. Let w be a smooth function on $M^{(j)}$, and let $\lambda \in [-1, 1]$. We say that w is a solution of $(\star_{\lambda,j})$ if w solves the PDE

$$g^{ik}\, g^{jl}\, \big(g_{ij} - (1 + |dw|_g^2)^{-1}\, \partial_i w\, \partial_j w\big)\, \big((1 + |dw|_g^2)^{-\frac{1}{2}}\, D^2_{k,l} w - \lambda\, q_{kl}\big) = \tau^2\, \zeta^2\, w$$

in $M^{(j)}$ with Dirichlet boundary condition $w = 0$ on $\partial M^{(j)}$.

The PDE above is different from the PDE studied in [19] and [7] in that we include the cutoff function ζ^2 on the right hand side of the equation.

In the following, we prove the existence of a solution of $(\star_{1,j})$. The proof is standard, and follows the arguments of Schoen-Yau [19], Eichmair [5], Eichmair-Metzger [9], and Korevaar-Simon [13]. We first establish C^0-estimates for solutions of $(\star_{\lambda,j})$.

Lemma 2.4. *Suppose that w is a solution of $(\star_{\lambda,j})$ for some $\lambda \in [-1, 1]$. Then $|w| \leq b(r) - b(r_j)$ on the domain $\{r_0 < r \leq r_j\}$.*

Proof. As above, we define a positive function Υ on the domain $\{r > r_0\}$ by $\Upsilon = b \circ r$, where r denotes the radial coordinate near infinity. It suffices to show that

$$\sup_{\{r_0 < r \leq r_j\}} (w - \Upsilon) \leq -b(r_j) \tag{6}$$

and

$$\inf_{\{r_0 < r \leq r_j\}} (w + \Upsilon) \geq b(r_j). \tag{7}$$

To prove the inequality (6), we argue by contradiction. Suppose that

$$\sup_{\{r_0<r\leq r_j\}} (w-\Upsilon) > -b(r_j).$$

Since $\lim_{s\searrow r_0} b'(s) = -\infty$, we can find a point $\bar{x} \in \{r_0 < r \leq r_j\}$ such that

$$w(\bar{x}) - \Upsilon(\bar{x}) = \sup_{\{r_0<r\leq r_j\}} (w-\Upsilon) > -b(r_j). \tag{8}$$

Using the Dirichlet boundary condition for w, we obtain $w - \Upsilon = -b(r_j)$ on $\partial M^{(j)} = \{r = r_j\}$. Using (8), we deduce that $\bar{x} \in \{r_0 < r < r_j\}$. Consequently, $dw = d\Upsilon$ and $D^2 w \leq D^2\Upsilon$ at the point $\bar{x}$. This implies

$$\begin{aligned} &g^{ik}\, g^{jl}\, \big(g_{ij} - (1+|dw|_g^2)^{-1}\, \partial_i w\, \partial_j w\big)\, \big((1+|dw|_g^2)^{-\frac{1}{2}}\, D^2_{k,l} w - \lambda\, q_{kl}\big) \\ &\leq g^{ik}\, g^{jl}\, \big(g_{ij} - (1+|d\Upsilon|_g^2)^{-1}\, \partial_i \Upsilon\, \partial_j \Upsilon\big)\, \big((1+|d\Upsilon|_g^2)^{-\frac{1}{2}}\, D^2_{k,l} \Upsilon - \lambda\, q_{kl}\big) \end{aligned}$$

at the point $\bar{x}$. The expression on the right hand side is strictly negative by Proposition 2.1. Since w is a solution of $(\star_{\lambda,j})$, it follows that $\tau^2\,\zeta^2\, w(\bar{x}) < 0$. Therefore, $w(\bar{x}) < 0$. Since $\Upsilon(\bar{x}) \geq b(r_j)$, we conclude that $w(\bar{x}) - \Upsilon(\bar{x}) < -b(r_j)$. This contradicts (8). This proves (6). The inequality (7) follows by replacing w by $-w$ and λ by $-\lambda$. This completes the proof of Lemma 2.4.

Remark 2.5. The estimate $|w| \leq b(r) - b(r_j)$ in Lemma 2.4 holds with equality along the boundary $\partial M^{(j)}$. In particular, $|dw|_g \leq |b'(r_j)|\, |dr|_g$ at each point on the boundary $\partial M^{(j)}$.

Lemma 2.6. *Suppose that w is a solution of $(\star_{\lambda,j})$ for some $\lambda \in [-1,1]$. Then $|w| \leq 2\, r_0^{n-2}\, r^{3-n}$ on the domain $\{2r_0 < r \leq r_j\}$.*

Proof. This follows by combining Proposition 2.2 and Lemma 2.4.

Lemma 2.7. *Suppose that w is a solution of $(\star_{\lambda,j})$ for some $\lambda \in [-1,1]$. Then*

$$\sup_{M^{(j)}} |w| \leq \max\Big\{2^{4-n}\, r_0, \tau^{-2} \sup_M n\, |q|_g\Big\}.$$

Proof. It suffices to show that

$$\sup_{M^{(j)}} w \leq \max\Big\{2^{4-n}\, r_0, \tau^{-2} \sup_M n\, |q|_g\Big\}, \tag{9}$$

and

$$\inf_{M^{(j)}} w \geq -\max\Big\{2^{4-n}\, r_0, \tau^{-2} \sup_M n\, |q|_g\Big\}. \tag{10}$$

To prove the inequality (9), we argue by contradiction. Suppose that

$$\sup_{M^{(j)}} w > \max\Big\{2^{4-n}\, r_0, \tau^{-2} \sup_M n\, |q|_g\Big\}.$$

We can find a point $\bar{x} \in M^{(j)}$ such that

$$w(\bar{x}) = \sup_{M^{(j)}} w > \max\Big\{2^{4-n}\, r_0, \tau^{-2} \sup_M n\, |q|_g\Big\}. \tag{11}$$

Using Lemma 2.6, we obtain $w \leq 2^{4-n}\, r_0$ on the domain $\{2r_0 < r \leq r_j\}$. Using (11), we deduce that $\bar{x} \in M^{(j)} \setminus \{2r_0 < r \leq r_j\}$. Consequently, $\zeta = 1$ at the point $\bar{x}$. Moreover, $dw = 0$ and $D^2 w \leq 0$ at the point $\bar{x}$. This implies

$$g^{ik}\, g^{jl}\, \big(g_{ij} - (1 + |dw|_g^2)^{-1}\, \partial_i w\, \partial_j w\big)\, \big((1 + |dw|_g^2)^{-\frac{1}{2}}\, D^2_{k,l} w - \lambda\, q_{kl}\big) \leq n\, |q|_g$$

at the point $\bar{x}$. Since w is a solution of $(\star_{\lambda,j})$, it follows that $\tau^2\, w \leq n\, |q|_g$ at the point $\bar{x}$. This contradicts (11). This proves (9). The inequality (10) follows by replacing w by $-w$ and λ by $-\lambda$. This completes the proof of Lemma 2.7.

In the next step, we establish C^1-estimates for solutions of $(\star_{\lambda,j})$.

Lemma 2.8. *Suppose that w is a solution of $(\star_{\lambda,j})$ for some $\lambda \in [-1, 1]$. Then $|dw|_g \leq C$ in the region $\{\frac{1}{2}\, r_j < r \leq r_j\}$. Here, C is a large constant that is independent of j.*

Proof. Let $d_{\bar{g}}$ denote the distance function with respect to the Euclidean metric $\bar{g}$. Given a point p in the region $\{\frac{1}{2}\, r_j < r < r_j\}$, we consider the Euclidean ball $\mathcal{B}_{\bar{g}}(p, \sigma)$, where $\sigma = \frac{1}{2}\, d_{\bar{g}}(p, \partial M^{(j)}) = \frac{1}{2}\, (r_j - r(p))$. Lemma 2.4 implies that $\sup_{\mathcal{B}_{\bar{g}}(p,\sigma)} |w| \leq C_0 \sigma$ for some uniform constant C_0. We now apply Proposition A.1 with $\Omega = \mathcal{B}_{\bar{g}}(p, \sigma)$ and

$$\psi = 2C_0 \sigma^{-1}\, (2\, d_{\bar{g}}(p, \cdot)^2 - \sigma^2).$$

Thus, we conclude that $|dw|_g \leq C$ at the point p, where C is a large constant that is independent of σ and j. This completes the proof of Lemma 2.8.

Lemma 2.9. *Suppose that w is a solution of $(\star_{\lambda,j})$ for some $\lambda \in [-1, 1]$. Then $|dw|_g \leq C$ in the region $M^{(j)} \setminus \{\frac{1}{2}\, r_j < r \leq r_j\}$. Here, C is a large constant that is independent of j.*

Proof. Let us fix a positive real number σ such that $\sigma < \frac{1}{2}\, \mathrm{inj}(M, g)$. It follows from Lemma 2.7 that $\sup_{M^{(j)}} |w| \leq C_1 \sigma$, where

$$C_1 = \sigma^{-1}\, \max\Big\{2^{4-n}\, r_0, \tau^{-2} \sup_M n\, |q|_g\Big\}.$$

We now apply Proposition A.1 with $\Omega = \mathcal{B}_g(p, \sigma)$ and

$$\psi = 2C_1 \sigma^{-1}\, (2\, d_g(p, \cdot)^2 - \sigma^2).$$

Thus, we conclude that $|dw|_g \leq C$ at the point p, where C is a large constant that is independent of j. This completes the proof of Lemma 2.9.

Proposition 2.10. *For each j, the boundary value problem $(\star_{1,j})$ has a smooth solution $u^{(j)}$. Moreover, $|u^{(j)}| \leq 2\, r_0^{n-2}\, r^{3-n}$ on the domain $\{2r_0 < r \leq r_j\}$.*

Proof. Let us fix an integer j. Lemma 2.7 gives a C^0-estimate for solutions of $(\star_{\lambda,j})$. Using Lemma 2.8 and Lemma 2.9, we obtain a C^1-estimate up to the boundary of $M^{(j)}$ for solutions of $(\star_{\lambda,j})$. Standard elliptic theory (see e.g. Theorem 13.7 in [10]) gives bounds for the higher derivatives of solutions of $(\star_{\lambda,j})$. Therefore, the set

$$\{\lambda \in [-1,1] : (\star_{\lambda,j}) \text{ has a solution}\}$$

is a closed subset of $[-1,1]$. On the other hand, it follows from Theorem 6.14 in [10] that, for every solution of $(\star_{\lambda,j})$, the linearized operator is invertible. Consequently, the set

$$\{\lambda \in [-1,1] : (\star_{\lambda,j}) \text{ has a solution}\}$$

is an open subset of $[-1,1]$. Since the function $w = 0$ is a solution of $(\star_{0,j})$, we conclude that $(\star_{\lambda,j})$ has a solution for each $\lambda \in [-1,1]$. This completes the proof of Proposition 2.10.

Proposition 2.11. *We can find a smooth function u on M such that*

$$g^{ik}\, g^{jl}\, \big(g_{ij} - (1+|du|_g^2)^{-1}\, \partial_i u\, \partial_j u\big)\, \big((1+|du|_g^2)^{-\frac{1}{2}}\, D^2_{k,l} u - q_{kl}\big) = \tau^2\, \zeta^2\, u$$

at each point in M and $|u| \leq 2\, r_0^{n-2}\, r^{3-n}$ on the domain $\{r > 2r_0\}$.

Proof. By Lemma 2.7, the sequence $u^{(j)}$ is uniformly bounded in C^0. Using Lemma 2.9, we obtain uniform C^1-bounds on every compact subset of M. Standard elliptic theory (see e.g. Theorem 13.6 in [10]) now gives uniform bounds for the higher derivatives of $u^{(j)}$ on every compact subset of M. Hence, after passing to a subsequence if necessary, the sequence $u^{(j)}$ converges in C^∞_{loc} to a smooth function u. It is easy to see that u has all the required properties. This completes the proof of Proposition 2.11.

Proposition 2.12. *For every positive integer m, the m-th order derivatives of the function u satisfy the bound*

$$|\bar{D}^m u|_{\bar{g}} \leq C(m)\, r^{3-n-m}$$

on the domain $\{r > 32r_0\}$.

Proof. It follows from Lemma 2.9 that $|du|_g \leq C$ for some uniform constant C. Standard elliptic theory (see e.g. Theorem 13.6 in [10]) now implies that

$$|\bar{D}^m u|_{\bar{g}} \leq C(m)\, r^{1-m}$$

on the domain $\{r > 8r_0\}$. On the domain $\{r > 16r_0\}$, we have $|u| \leq 2\, r_0^{n-2}\, r^{3-n}$ and $|\bar{D}^m q|_{\bar{g}} \leq C(m)\, r^{1-n-m}$ for every nonnegative integer m. Hence, standard interior estimates for linear PDE imply that

$$|\bar{D}^m u|_{\bar{g}} \leq C(m)\, r^{3-n-m}$$

on the domain $\{r > 32r_0\}$. This completes the proof of Proposition 2.12.

2.4. **The metric $\breve{g}$.** Let u denote the function constructed in Proposition 2.11. We define a smooth Riemannian metric $\breve{g}$ and a 1-form Ξ on M by

$$\breve{g}_{ij} = g_{ij} + \partial_i u\, \partial_j u$$

and

$$\Xi_k = \frac{1}{2}\, \partial_k \log(1 + |du|_g^2) - (1 + |du|_g^2)^{-\frac{1}{2}}\, g^{ij}\, q_{ik}\, \partial_j u.$$

Lemma 2.13. *We have*

$$|\bar{D}^m(\breve{g} - (1 + \alpha\, r^{2-n})\, \bar{g})|_{\bar{g}} \leq O(r^{2-n-m-2\delta})$$

for every nonnegative integer m. Moreover, $|R_{\breve{g}}| \leq O(r^{-n-2\delta})$, $|\Xi|_{\breve{g}} \leq O(r^{3-2n})$, and $|\mathrm{div}_{\breve{g}}\Xi| \leq O(r^{2-2n})$. In particular, the functions $|R_{\breve{g}}|$, $|\Xi|_{\breve{g}}^2$, and $|\mathrm{div}_{\breve{g}}\Xi|$ are integrable.

Proof. Using Proposition 2.12 together with the identity $\breve{g} - g = du \otimes du$, we obtain

$$|\bar{D}^m(\breve{g} - g)|_{\bar{g}} \leq O(r^{4-2n-m})$$

for every nonnegative integer m. Moreover,

$$|\bar{D}^m(g - (1 + \alpha\, r^{2-n})\, \bar{g})|_{\bar{g}} \leq O(r^{2-n-m-2\delta})$$

for every nonnegative integer m. Putting these facts together, we conclude that

$$|\bar{D}^m(\breve{g} - (1 + \alpha\, r^{2-n})\, \bar{g})|_{\bar{g}} \leq O(r^{2-n-m-2\delta})$$

for every nonnegative integer m. This implies $|R_{\breve{g}}| \leq O(r^{-n-2\delta})$. Moreover, using Proposition 2.12 and the estimate $|q|_{\bar{g}} \leq O(r^{1-n})$, we obtain $|\Xi|_{\bar{g}} \leq O(r^{3-2n})$ near infinity. The estimate $|\mathrm{div}_{\breve{g}}\Xi| \leq O(r^{2-2n})$ follows similarly. This completes the proof of Lemma 2.13.

Proposition 2.14. *We have*

$$\frac{1}{2}\, R_{\breve{g}} - |\Xi|_{\breve{g}}^2 + \mathrm{div}_{\breve{g}}\Xi \geq Q + (\kappa_0^2 - \tau^2\, u^2)\, |d\zeta|_g^2 + (\kappa_1 - \tau^2\, |u|)\, \zeta^2\, n\, |q|_g$$

at each point in M.

Proof. Applying Proposition B.1 with $w = u$ and $\Theta = \tau^2\,\zeta^2\,u$ gives

$$\begin{aligned}
&\frac{1}{2}\,R_{\breve{g}} - |\Xi|_{\breve{g}}^2 + \mathrm{div}_{\breve{g}}\Xi \\
&= \frac{1}{2}\,\breve{g}^{ik}\,\breve{g}^{jl}\,\big((1+|du|_g^2)^{-\frac{1}{2}}\,D^2_{i,j}u - q_{ij}\big)\,\big((1+|du|_g^2)^{-\frac{1}{2}}\,D^2_{k,l}u - q_{kl}\big) \\
&+ \mu - (1+|du|_g^2)^{-\frac{1}{2}}\,\langle du, J\rangle_g \\
&+ \tau^2\,(1+|du|_g^2)^{-\frac{1}{2}}\,(|\zeta\,du + u\,d\zeta|_g^2 - u^2\,|d\zeta|_g^2) + \frac{1}{2}\,\tau^4\,\zeta^4\,u^2 + \tau^2\,\zeta^2\,u\,\mathrm{tr}_{\breve{g}}(q)
\end{aligned}$$

at each point in M. Using the inequality $|\mathrm{tr}_{\breve{g}}(q)| \leq n\,|q|_g$, we obtain

$$\frac{1}{2}\,R_{\breve{g}} - |\Xi|_{\breve{g}}^2 + \mathrm{div}_{\breve{g}}\Xi \geq \mu - |J|_g - \tau^2\,u^2\,|d\zeta|_g^2 - \tau^2\,\zeta^2\,|u|\,n\,|q|_g \tag{12}$$

at each point in M. On the other hand, the inequality (2) gives

$$\begin{aligned}
&\mu - |J|_g - \tau^2\,u^2\,|d\zeta|_g^2 - \tau^2\,\zeta^2\,|u|\,n\,|q|_g \\
&\geq Q + (\kappa_0^2 - \tau^2\,u^2)\,|d\zeta|_g^2 + (\kappa_1 - \tau^2\,|u|)\,\zeta^2\,n\,|q|_g
\end{aligned} \tag{13}$$

at each point in M. If we combine (12) and (13), the assertion follows. This completes the proof of Proposition 2.14.

Corollary 2.15. *Suppose that f is a smooth test function with the property that f is supported in the domain $\big\{|u| < \min\{\kappa_0\,\tau^{-1}, \kappa_1\,\tau^{-2}\}\big\}$ and f is constant near infinity. Then*

$$\int_M |df|_{\breve{g}}^2\,d\mathrm{vol}_{\breve{g}} + \frac{1}{2}\int_M R_{\breve{g}}\,f^2\,d\mathrm{vol}_{\breve{g}} \geq \int_M Q\,f^2\,d\mathrm{vol}_{\breve{g}}.$$

Note that the integral $\int_M R_{\breve{g}}\,f^2\,d\mathrm{vol}_{\breve{g}}$ is well-defined in view of Lemma 2.13, and the integral $\int_M Q\,f^2\,d\mathrm{vol}_{\breve{g}}$ is well-defined in view of (3).

Proof. Proposition 2.14 implies that

$$\frac{1}{2}\,R_{\breve{g}} - |\Xi|_{\breve{g}}^2 + \mathrm{div}_{\breve{g}}\Xi \geq Q$$

on the domain $\big\{|u| < \min\{\kappa_0\,\tau^{-1}, \kappa_1\,\tau^{-2}\}\big\}$. Since f is supported in the domain $\big\{|u| < \min\{\kappa_0\,\tau^{-1}, \kappa_1\,\tau^{-2}\}\big\}$, we obtain

$$\int_M \Big(\frac{1}{2}\,R_{\breve{g}} - |\Xi|_{\breve{g}}^2 + \mathrm{div}_{\breve{g}}\Xi\Big)\,f^2\,d\mathrm{vol}_{\breve{g}} \geq \int_M Q\,f^2\,d\mathrm{vol}_{\breve{g}}. \tag{14}$$

Note that the integral on the left hand side is well-defined in view of Lemma 2.13, and the integral on the right hand side is well-defined in view of (3). Moreover, Lemma 2.13 implies that $|\Xi|_{\breve{g}} \leq O(r^{3-2n})$ near infinity. Since f is constant near infinity and $\breve{g}$ is uniformly equivalent to $\bar{g}$ near infinity, we conclude that

$$\int_M \mathrm{div}_{\breve{g}}(f^2\,\Xi)\,d\mathrm{vol}_{\breve{g}} = 0 \tag{15}$$

by the divergence theorem. Subtracting (15) from (14) gives

$$\int_M |df|_{\breve{g}}^2 \, d\mathrm{vol}_{\breve{g}} + \frac{1}{2} \int_M R_{\breve{g}} \, f^2 \, d\mathrm{vol}_{\breve{g}} - \int_M |df + f \, \Xi|_{\breve{g}}^2 \, d\mathrm{vol}_{\breve{g}}$$
$$\geq \int_M Q \, f^2 \, d\mathrm{vol}_{\breve{g}}.$$

This completes the proof of Corollary 2.15.

Lemma 2.16. *We have*

$$\mathcal{N}_{\breve{g}}(E_0, s_1 + 2s_0) \subset \Big\{ |u| \leq \frac{1}{2} \min\{\kappa_0 \, \tau^{-1}, \kappa_1 \, \tau^{-2}\} \Big\}.$$

Proof. Recall that

$$|u| \leq 2 \, r_0^{n-2} \, r^{3-n} \leq 2^{10-3n} \, r_0$$

on the domain $E_0 = \{r > 8r_0\}$. Moreover, $|du|_{\breve{g}} \leq 1$ by definition of the metric $\breve{g}$. Using (5), we conclude that

$$|u| \leq 2^{10-3n} \, r_0 + s_1 + 2s_0 \leq \frac{1}{2} \min\{\kappa_0 \, \tau^{-1}, \kappa_1 \, \tau^{-2}\}$$

on the set $\mathcal{N}_{\breve{g}}(E_0, s_1 + 2s_0)$. This completes the proof of Lemma 2.16.

Lemma 2.17. *We have*

$$\mathcal{N}_{\breve{g}}(E_0, 2s_0) \subset \mathcal{N}_g(E_0, 2s_0).$$

In particular,

$$Q(x) > \frac{128}{s_1 s_0}$$

for each point $x \in \mathcal{N}_{\breve{g}}(E_0, 2s_0) \setminus E_0$.

Proof. The first statement follows from the pointwise inequality $\breve{g} \geq g$. The second statement follows from the first statement together with (4). This completes the proof of Lemma 2.17.

In the next step, we recall an important lemma from the work of Lesourd-Unger-Yau.

Lemma 2.18 (cf. Lesourd-Unger-Yau [15], Proposition 3.1)**.** *We can find an open, connected domain E with smooth boundary, a smooth function Φ defined on E, and a smooth function $\hat{Q}$ defined on E with the following properties:*

- *The closure of E_0 is contained in E.*
- *The domain E is contained in $\mathcal{N}_{\breve{g}}(E_0, s_1 + 2s_0)$. In particular, the complement $E \setminus E_0$ is a bounded subset of (M, g).*
- *$\Phi = 0$ and $\hat{Q} = \frac{1}{2} Q$ at each point in E_0.*
- *$\Phi \leq 0$ and $\hat{Q} > 0$ at each point in E.*
- *$\Phi \to -\infty$ on the boundary ∂E.*

- $Q + \frac{1}{2}\,\Phi^2 - 2\,|d\Phi|_{\breve{g}} \geq 2\hat{Q}$ *at each point in* E.

Lemma 2.18 follows from Lemma 2.17. A detailed proof can be found in [2].

Remark 2.19. In Lemma 2.18, we allow the possibility that $\partial E = \emptyset$.

After these preparations, we now complete the proof of Theorem 1.1. Using Lemma 2.16 and Lemma 2.18, we obtain

$$E \subset \mathcal{N}_{\breve{g}}(E_0, s_1 + 2s_0) \subset \Big\{|u| \leq \frac{1}{2}\,\min\{\kappa_0\,\tau^{-1}, \kappa_1\,\tau^{-2}\}\Big\}$$

by Lemma 2.16 and Lemma 2.18. Consequently, we can find an open, connected domain $\hat{E}$ with smooth boundary such that the closure of E is contained in $\hat{E}$, and the closure of $\hat{E}$ is contained in the domain $\big\{|u| < \min\{\kappa_0\,\tau^{-1}, \kappa_1\,\tau^{-2}\}\big\}$. Using Corollary 2.15, we conclude that

$$\int_M |df|_{\breve{g}}^2\,d\mathrm{vol}_{\breve{g}} + \frac{1}{2}\int_M R_{\breve{g}}\,f^2\,d\mathrm{vol}_{\breve{g}} \geq \int_M Q\,f^2\,d\mathrm{vol}_{\breve{g}}$$

for every smooth test function f with the property that f is supported in $\hat{E}$ and f is constant near infinity. Finally, since $\alpha \leq 0$, Lemma 2.13 implies that $(M, \breve{g})$ has nonpositive mass.

We now follow the arguments in Subsections 3.2–3.8 in [2], with $\rho = 1$. In conclusion, we obtain an $(n-1)$-dataset (in the sense of Definition 1.3 in [2]) which has zero mass (in the sense of Definition 1.4 in [2]). This contradicts Theorem 1.5 in [2]. This completes the proof of Theorem 1.1.

Appendix A. A Korevaar-Simon-type estimate for the modified Jang equation

The following estimate goes back to work of Korevaar-Simon [13] and was extended to the Jang equation by Eichmair [5] and Eichmair-Metzger [9].

Proposition A.1 (cf. N. Korevaar, L. Simon [13])**.** *Let* $A \geq 4$ *and* $\sigma > 0$. *Let* (M, g) *be a Riemannian manifold of dimension* n, *and let* q *be a symmetric* $(0,2)$*-tensor on* M. *Let* Ω *be an open domain in* M *with compact closure, and let* $\zeta, \psi \in C(\bar{\Omega}) \cap C^\infty(\Omega)$. *We assume that* $\mathrm{Ric} \geq -A\,\sigma^{-2}\,g$, $n\,|q|_g \leq A\,\sigma^{-1}$, $n\,|Dq|_g \leq A\,\sigma^{-2}$, $|\psi| \leq \frac{1}{4}\,A\,\sigma$, $|d\psi|_g \leq \frac{1}{4}\,A$ *and* $n\,|D^2\psi|_g \leq \frac{1}{4}\,A\,\sigma^{-1}$ *at each point in* Ω. *Let* $w \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ *be a solution of the PDE*

$$(16) \qquad \big(g^{ij} - (1 + |dw|_g^2)^{-1}\,\partial^i w\,\partial^j w\big)\,\big((1 + |dw|_g^2)^{-\frac{1}{2}}\,D^2_{i,j} w - q_{ij}\big) = \zeta^2\,w$$

in Ω *satisfying* $w < \psi$ *along* $\partial\Omega$. *If*

$$(1 + \zeta^2\,\sigma^2 + |d\zeta|_g^2\,\sigma^4)\,|w| \leq A\,\sigma$$

at each point in Ω, *then*

$$(e^{A^2\sigma^{-1}(w-\psi)} - 1)\,(1 + |dw|_g^2)^{\frac{1}{2}} \leq C(A)$$

at each point in Ω, *where* $C(A)$ *is a constant that depends only on* A.

Proof. Let

$$\Lambda = \sup_\Omega (e^{A^2 \sigma^{-1} (w-\psi)} - 1)\,(1 + |dw|_g^2)^{\frac{1}{2}}.$$

If $\Lambda \leq 0$, we are done. It remains to consider the case $\Lambda > 0$. Since $w < \psi$ along $\partial\Omega$, we can find a point $\bar{x} \in \Omega$ where the function

$$(e^{A^2 \sigma^{-1} (w-\psi)} - 1)\,(1 + |dw|_g^2)^{\frac{1}{2}}$$

attains its maximum. Clearly,

$$e^{A^2 \sigma^{-1} (w-\psi)} - 1 \leq \Lambda\,(1 + |dw|_g^2)^{-\frac{1}{2}}$$

at each point in Ω, and equality holds at the point $\bar{x}$. For abbreviation, let

$$S^{ij} = 2\,q^{ij} - (1 + |dw|_g^2)^{-1}\,q^{kl}\,\partial_k w\,\partial_l w\,g^{ij} - (\mathrm{tr}_g(q) + \zeta^2\,w)\,g^{ij}.$$

Then

$$\begin{aligned}
&\big(g^{ij} - (1 + |dw|_g^2)^{-1}\,\partial^i w\,\partial^j w\big)\,D^2_{i,j}(e^{A^2 \sigma^{-1} (w-\psi)}) \\
&+ (1 + |dw|_g^2)^{-\frac{1}{2}}\,S^{ij}\,\partial_i w\,\partial_j(e^{A^2 \sigma^{-1} (w-\psi)}) \\
&\leq \Lambda\,\big(g^{ij} - (1 + |dw|_g^2)^{-1}\,\partial^i w\,\partial^j w\big)\,D^2_{i,j}\big((1 + |dw|_g^2)^{-\frac{1}{2}}\big) \\
&+ \Lambda\,(1 + |dw|_g^2)^{-\frac{1}{2}}\,S^{ij}\,\partial_i w\,\partial_j\big((1 + |dw|_g^2)^{-\frac{1}{2}}\big)
\end{aligned}$$

at the point $\bar{x}$. The identity (16) implies

$$\begin{aligned}
&\big(g^{ij} - (1 + |dw|_g^2)^{-1}\,\partial^i w\,\partial^j w)\,D^2_{i,j} w \\
&= -(1 + |dw|_g^2)^{-\frac{1}{2}}\,q^{ij}\,\partial_i w\,\partial_j w + (1 + |dw|_g^2)^{\frac{1}{2}}\,(\mathrm{tr}_g(q) + \zeta^2\,w)
\end{aligned} \tag{17}$$

at each point in Ω. Using the identity

$$\begin{aligned}
&\big(g^{ij} - (1 + |dw|_g^2)^{-1}\,\partial^i w\,\partial^j w\big)\,D^2_{i,j}\big((1 + |dw|_g^2)^{-\frac{1}{2}}\big) \\
&= -\big(g^{ij} - (1 + |dw|_g^2)^{-1}\,\partial^i w\,\partial^j w\big)\,\big(g^{kl} - (1 + |dw|_g^2)^{-1}\,\partial^k w\,\partial^l w\big) \\
&\qquad \cdot (1 + |dw|_g^2)^{-\frac{3}{2}}\,D^2_{i,k} w\,D^2_{j,l} w \\
&- (1 + |dw|_g^2)^{-\frac{3}{2}}\,\mathrm{Ric}^{kl}\,\partial_k w\,\partial_l w \\
&- (1 + |dw|_g^2)^{-\frac{3}{2}}\,\partial^k w\,\partial_k\big(\big(g^{ij} - (1 + |dw|_g^2)^{-1}\,\partial^i w\,\partial^j w\big)\,D^2_{i,j} w\big)
\end{aligned}$$

together with (17), we obtain

$$
\begin{aligned}
&\big(g^{ij} - (1 + |dw|_g^2)^{-1}\, \partial^i w\, \partial^j w\big)\, D^2_{i,j}\big((1 + |dw|_g^2)^{-\frac{1}{2}}\big) \\
&+ (1 + |dw|_g^2)^{-\frac{1}{2}}\, S^{ij}\, \partial_i w\, \partial_j\big((1 + |dw|_g^2)^{-\frac{1}{2}}\big) \\
&= -\big(g^{ij} - (1 + |dw|_g^2)^{-1}\, \partial^i w\, \partial^j w\big)\, \big(g^{kl} - (1 + |dw|_g^2)^{-1}\, \partial^k w\, \partial^l w\big) \\
&\qquad \cdot (1 + |dw|_g^2)^{-\frac{3}{2}}\, D^2_{i,k} w\, D^2_{j,l} w \\
&- (1 + |dw|_g^2)^{-\frac{3}{2}}\, \mathrm{Ric}^{kl}\, \partial_k w\, \partial_l w \\
&- (1 + |dw|_g^2)^{-1}\, D^k q^{ij}\, \big(g_{ij} - (1 + |dw|_g^2)^{-1}\, \partial_i w\, \partial_j w\big)\, \partial_k w \\
&- (1 + |dw|_g^2)^{-1}\, (|\zeta\, dw + w\, d\zeta|_g^2 - w^2\, |d\zeta|_g^2)
\end{aligned}
$$

at each point in Ω. Using the estimate $\mathrm{Ric} \geq -A\, \sigma^{-2}\, g$, we conclude that

$$
\begin{aligned}
&\Lambda\, \big(g^{ij} - (1 + |dw|_g^2)^{-1}\, \partial^i w\, \partial^j w\big)\, D^2_{i,j}\big((1 + |dw|_g^2)^{-\frac{1}{2}}\big) \\
&+ \Lambda\, (1 + |dw|_g^2)^{-\frac{1}{2}}\, S^{ij}\, \partial_i w\, \partial_j\big((1 + |dw|_g^2)^{-\frac{1}{2}}\big) \\
&\leq \Lambda\, A\, \sigma^{-2}\, (1 + |dw|_g^2)^{-\frac{3}{2}}\, |dw|_g^2 \\
&+ \Lambda\, (1 + |dw|_g^2)^{-1}\, n\, |Dq|_g\, |dw|_g \\
&+ \Lambda\, (1 + |dw|_g^2)^{-1}\, w^2\, |d\zeta|_g^2 \\
&\leq A\, \sigma^{-2}\, e^{A^2 \sigma^{-1} (w - \psi)}\, (1 + |dw|_g^2)^{-1}\, |dw|_g^2 \\
&+ e^{A^2 \sigma^{-1} (w - \psi)}\, (1 + |dw|_g^2)^{-\frac{1}{2}}\, n\, |Dq|_g\, |dw|_g \\
&+ e^{A^2 \sigma^{-1} (w - \psi)}\, (1 + |dw|_g^2)^{-\frac{1}{2}}\, w^2\, |d\zeta|_g^2
\end{aligned}
$$

at the point $\bar{x}$. On the other hand, using the identity (17), we obtain

$$\begin{aligned}
&\big(g^{ij} - (1 + |dw|_g^2)^{-1}\, \partial^i w\, \partial^j w\big)\, D_{i,j}^2 (e^{A^2 \sigma^{-1} (w-\psi)}) \\
&+ (1 + |dw|_g^2)^{-\frac{1}{2}}\, S^{ij}\, \partial_i w\, \partial_j (e^{A^2 \sigma^{-1} (w-\psi)}) \\
&= A^4\, \sigma^{-2}\, e^{A^2 \sigma^{-1} (w-\psi)}\, \big(g^{ij} - (1 + |dw|_g^2)^{-1}\, \partial^i w\, \partial^j w\big)\, \partial_i (w-\psi)\, \partial_j (w-\psi) \\
&+ A^2\, \sigma^{-1}\, e^{A^2 \sigma^{-1} (w-\psi)}\, \big(g^{ij} - (1 + |dw|_g^2)^{-1}\, \partial^i w\, \partial^j w\big)\, D_{i,j}^2 (w-\psi) \\
&+ A^2\, \sigma^{-1}\, e^{A^2 \sigma^{-1} (w-\psi)}\, (1 + |dw|_g^2)^{-\frac{1}{2}}\, S^{ij}\, \partial_i w\, \partial_j (w-\psi) \\
&= A^4\, \sigma^{-2}\, e^{A^2 \sigma^{-1} (w-\psi)}\, (1 + |dw|_g^2)^{-1}\, |dw|_g^2 \\
&- 2A^4\, \sigma^{-2}\, e^{A^2 \sigma^{-1} (w-\psi)}\, (1 + |dw|_g^2)^{-1}\, \langle dw, d\psi \rangle_g \\
&+ A^4\, \sigma^{-2}\, e^{A^2 \sigma^{-1} (w-\psi)}\, \big(g^{ij} - (1 + |dw|_g^2)^{-1}\, \partial^i w\, \partial^j w\big)\, \partial_i \psi\, \partial_j \psi \\
&- A^2\, \sigma^{-1}\, e^{A^2 \sigma^{-1} (w-\psi)}\, \big(g^{ij} - (1 + |dw|_g^2)^{-1}\, \partial^i w\, \partial^j w\big)\, D_{i,j}^2 \psi \\
&- 2A^2\, \sigma^{-1}\, e^{A^2 \sigma^{-1} (w-\psi)}\, (1 + |dw|_g^2)^{-\frac{1}{2}}\, q^{ij}\, \partial_i w\, \partial_j \psi \\
&+ A^2\, \sigma^{-1}\, e^{A^2 \sigma^{-1} (w-\psi)}\, (1 + |dw|_g^2)^{-\frac{3}{2}}\, q^{ij}\, \partial_i w\, \partial_j w\, (1 + \langle dw, d\psi \rangle_g) \\
&+ A^2\, \sigma^{-1}\, e^{A^2 \sigma^{-1} (w-\psi)}\, (1 + |dw|_g^2)^{-\frac{1}{2}}\, (\mathrm{tr}_g(q) + \zeta^2\, w)\, (1 + \langle dw, d\psi \rangle_g)
\end{aligned}$$

at each point in Ω. Consequently,

$$\begin{aligned}
&\big(g^{ij} - (1 + |dw|_g^2)^{-1}\, \partial^i w\, \partial^j w\big)\, D_{i,j}^2 (e^{A^2 \sigma^{-1} (w-\psi)}) \\
&+ (1 + |dw|_g^2)^{-\frac{1}{2}}\, S^{ij}\, \partial_i w\, \partial_j (e^{A^2 \sigma^{-1} (w-\psi)}) \\
&\geq A^4\, \sigma^{-2}\, e^{A^2 \sigma^{-1} (w-\psi)}\, (1 + |dw|_g^2)^{-1}\, |dw|_g^2 \\
&- 2A^4\, \sigma^{-2}\, e^{A^2 \sigma^{-1} (w-\psi)}\, (1 + |dw|_g^2)^{-1}\, |dw|_g\, |d\psi|_g \\
&- A^2\, \sigma^{-1}\, e^{A^2 \sigma^{-1} (w-\psi)}\, n\, |D^2 \psi|_g \\
&- A^2\, \sigma^{-1}\, e^{A^2 \sigma^{-1} (w-\psi)}\, (1 + |dw|_g^2)^{-\frac{1}{2}}\, (2n\, |q|_g + \zeta^2\, |w|)\, (1 + |dw|_g\, |d\psi|_g)
\end{aligned}$$

at each point in Ω. Putting these facts together, we obtain

$$\begin{aligned}
&A^4\, |dw|_g^2 - 2A^4\, |dw|_g\, |d\psi|_g - A^2\, \sigma\, (1 + |dw|_g^2)\, n\, |D^2 \psi|_g \\
&- A^2\, \sigma\, (1 + |dw|_g^2)^{\frac{1}{2}}\, (2n\, |q|_g + \zeta^2\, |w|)\, (1 + |dw|_g\, |d\psi|_g) \\
&\leq A\, |dw|_g^2 + \sigma^2\, (1 + |dw|_g^2)^{\frac{1}{2}}\, n\, |Dq|_g\, |dw|_g \\
&+ \sigma^2\, (1 + |dw|_g^2)^{\frac{1}{2}}\, w^2\, |d\zeta|_g^2
\end{aligned}$$

at the point $\bar{x}$. In order to estimate the terms on the left hand side, we use the inequalities $|d\psi|_g \leq \frac{1}{4}\, A$, $n\, |D^2 \psi|_g \leq \frac{1}{4}\, A\, \sigma^{-1}$, $n\, |q|_g \leq A\, \sigma^{-1}$, and $\zeta^2\, |w| \leq A\, \sigma^{-1}$. In order to estimate the terms on the right hand side, we use the inequalities $n\, |Dq|_g \leq A\, \sigma^{-2}$, $|w| \leq A\, \sigma$, and $|d\zeta|_g^2\, |w| \leq A\, \sigma^{-3}$. Thus, we conclude that

$$A^4 \, |dw|_g^2 - \frac{1}{2} \, A^5 \, |dw|_g - \frac{1}{4} \, A^3 \, (1 + |dw|_g^2) - 3A^3 \, (1 + |dw|_g^2)^{\frac{1}{2}} \, (1 + \frac{1}{4} \, A \, |dw|_g)$$
$$\leq A \, |dw|_g^2 + A \, (1 + |dw|_g^2)^{\frac{1}{2}} \, |dw|_g + A^2 \, (1 + |dw|_g^2)^{\frac{1}{2}}$$

at the point $\bar{x}$. Since $\frac{1}{4} \, A^4 - \frac{1}{4} \, A^3 > 2A$, we can bound $|dw(\bar{x})|_g$ from above by some function of A. Since $|\psi| \leq A \, \sigma$ and $|w| \leq A \, \sigma$, we conclude that

$$(e^{A^2 \sigma^{-1} (w - \psi)} - 1) \, (1 + |dw|_g^2)^{\frac{1}{2}} \leq e^{2A^3} \, (1 + |dw|_g^2)^{\frac{1}{2}} \leq C(A)$$

at the point $\bar{x}$. Thus, $\Lambda \leq C(A)$, as claimed. This completes the proof of Proposition A.1.

Appendix B. The Schoen-Yau identity for the modified Jang equation

In this section, we state a crucial identity which originated in the groundbreaking work of Schoen and Yau [19]. The following identity is a straightforward generalization of the identity proved in the work of Jang [12] and Schoen and Yau [19].

Proposition B.1 (cf. P.S. Jang [12]; R. Schoen, S.T. Yau [19])**.** *Let (M, g) be a Riemannian manifold, and let q be a symmetric $(0, 2)$-tensor on M. Suppose that w and Θ are smooth functions on M such that*

$$g^{ik} \, g^{jl} \, \big(g_{ij} - (1 + |dw|_g^2)^{-1} \, \partial_i w \, \partial_j w\big) \, \big((1 + |dw|_g^2)^{-\frac{1}{2}} \, D^2_{k,l} w - q_{kl}\big) = \Theta$$

at each point in M. We define a Riemannian metric $\breve{g}$ and a one-form Ξ by

$$\breve{g}_{ij} = g_{ij} + \partial_i w \, \partial_j w$$

and

$$\Xi_k = \frac{1}{2} \, \partial_k \log(1 + |dw|_g^2) - (1 + |dw|_g^2)^{-\frac{1}{2}} \, g^{ij} \, q_{ik} \, \partial_j w.$$

Then

$$\begin{aligned} &\frac{1}{2} \, R_{\breve{g}} - |\Xi|_{\breve{g}}^2 + \mathrm{div}_{\breve{g}} \Xi \\ &= \frac{1}{2} \, \breve{g}^{ik} \, \breve{g}^{jl} \, \big((1 + |dw|_g^2)^{-\frac{1}{2}} \, D^2_{i,j} w - q_{ij}\big) \, \big((1 + |dw|_g^2)^{-\frac{1}{2}} \, D^2_{k,l} w - q_{kl}\big) \\ &+ \mu - (1 + |dw|_g^2)^{-\frac{1}{2}} \, \langle dw, J \rangle_g \\ &+ (1 + |dw|_g^2)^{-\frac{1}{2}} \, \langle dw, d\Theta \rangle_g + \frac{1}{2} \, \Theta^2 + \Theta \, \mathrm{tr}_{\breve{g}}(q) \end{aligned}$$

at each point in M.

Proof. This identity can be derived by applying Proposition 7.32 in [14] to the graph of w in $M \times \mathbb{R}$. We refer to Subsection 3.6 in [1] and Section 2 in [7] for further discussion.

Remark B.2. In the work of Schoen and Yau [19], the identity in Proposition B.1 is applied with $\Theta = 0$. A more general version including a capillary term was considered in Eichmair's work (see [5], p. 569–570).

References

[1] L. Andersson, M. Eichmair, and J. Metzger, *Jang's equation and its applications to marginally trapped surfaces,* Complex Analysis and Dynamical Systems IV, Contemp. Math. vol. 554, pp. 13–45, Amer. Math. Soc., Providence RI (2011)

[2] S. Brendle and Y. Wang, *A dimension descent scheme for the positive mass theorem in arbitrary dimension,* arXiv:2604.08473

[3] D. Christodoulou and N. O'Murchadha, *The boost problem in general relativity,* Comm. Math. Phys. 80, 271–300 (1981)

[4] P.T. Chruściel and E. Delay, *The hyperbolic positive energy theorem,* arXiv:1901.05263, to appear in J. Eur. Math. Soc.

[5] M. Eichmair, *The Plateau problem for marginally outer trapped surfaces,* J. Diff. Geom. 83, 551–584 (2009)

[6] M. Eichmair, *Existence, regularity, and properties of generalized apparent horizons,* Comm. Math. Phys. 294, 745–760 (2010)

[7] M. Eichmair, *The Jang equation reduction of the spacetime positive energy theorem in dimensions less than eight,* Comm. Math. Phys. 319, 575–593 (2013)

[8] M. Eichmair, L.H. Huang, D.A. Lee, and R. Schoen, *The spacetime positive mass theorem in dimensions less than eight,* J. Eur. Math. Soc. 18, 83–121 (2016)

[9] M. Eichmair and J. Metzger, *Jenkins-Serrin-type results for the Jang equation,* J. Diff. Geom. 102, 207–242 (2016)

[10] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag (2001)

[11] L.-H. Huang and D. A. Lee, *Equality in the spacetime positive mass theorem*, Comm. Math. Phys. 376, 2379–2407 (2020)

[12] P.S. Jang, *On the positivity of energy in general relativity,* J. Math. Phys. 19, 1152–1155 (1978)

[13] N.J. Korevaar and L.M. Simon, *Continuity estimates for solutions to the prescribed-curvature Dirichlet problem,* Math. Z. 197, 457–464 (1987)

[14] D.A. Lee, *Geometric relativity,* Graduate Studies in Mathematics, vol. 201, Amer. Math. Soc., Providence RI (2019)

[15] M. Lesourd, R. Unger, and S.T. Yau, *The positive mass theorem with arbitrary ends,* J. Diff. Geom. 128, 257–293 (2024)

[16] J. Lohkamp, *The higher dimensional positive mass theorem II,* arXiv:1612.07505

[17] R. Schoen and S.T. Yau, *On the proof of the positive mass conjecture in general relativity,* Comm. Math. Phys. 65, 45–76 (1979)

[18] R. Schoen and S.T. Yau, *On the structure of manifolds with positive scalar curvature,* Manuscripta Math. 28, 159–183 (1979)

[19] R. Schoen and S.T. Yau, *Proof of the positive mass theorem II,* Comm. Math. Phys. 79, 231–260 (1981)

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