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# Solving dynamic general equilibrium models using a second-order approximation to the policy function

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## Abstract

This paper derives a second-order approximation to the solution of a general class of discrete-time rational expectations models. The main theoretical contribution is to show that for any model belonging to that class, the coefficients on the terms linear and quadratic in the state vector in a second-order expansion of the decision rule are independent of the volatility of the exogenous shocks. In addition, the paper presents a set of MATLAB programs that implement the proposed second-order approximation method and applies it to a number of model economies. © 2003 Elsevier B.V. All rights reserved.

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## 1. Introduction

Since the seminal papers of Kydland and Prescott (1982) and King et al. (1988), it has become commonplace in macroeconomics to approximate the solution to non-linear, dynamic, stochastic, general equilibrium models using linear methods. Linear approximation methods are useful to characterize certain aspects of the dynamic properties of complicated models. In particular, if the support of the shocks driving aggregate fluctuations is small and an interior stationary solution exists, first-order approximations provide adequate answers to questions such as local existence and determinacy of equilibrium and the size of the second moments of endogenous variables.

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However, first-order approximation techniques are not well suited to handle questions such as welfare comparisons across alternative stochastic or policy environments. For example, [Kim and Kim \(in press\)](#) show that in a simple two-agent economy, a welfare comparison based on an evaluation of the utility function using a linear approximation to the policy function may yield the spurious result that welfare is higher under autarky than under full risk sharing. The problem here is that some second- and higher-order terms of the equilibrium welfare function are omitted while others are included. Consequently, the resulting criterion is inaccurate to order two or higher. The same problem arises under the common practice in macroeconomics of evaluating a second-order approximation to the objective function using a first-order approximation to the decision rules. For in this case, too, some second-order terms of the equilibrium welfare function are ignored while others are not.<sup>1</sup> In general, a correct second-order approximation of the equilibrium welfare function requires a second-order approximation to the policy function.

In this paper, we derive a second-order approximation to the policy function of a general class of dynamic, discrete-time, rational expectations models. A strength of our approach is not to follow a value function formulation. This allows us to tackle easily a wide variety of model economies that do not lend themselves naturally to the value function specification. To obtain an accurate second-order approximation, we use a perturbation method that incorporates a scale parameter for the standard deviations of the exogenous shocks as an argument of the policy function. In approximating the policy function, we take a second-order Taylor expansion with respect to the state variables as well as this scale parameter. This technique was formally introduced by [Fleming \(1971\)](#) and has been applied extensively to economic models by [Judd and co-authors](#) (see [Judd, 1998](#), and the references cited therein).

The main theoretical contributions of the paper are: First, it shows analytically that in general the first derivative of the policy function with respect to the parameter scaling the variance/covariance matrix of the shocks is zero at the steady state regardless of whether the model displays the certainty-equivalence property or not.<sup>2</sup> Second, it proves that in general the cross derivative of the policy function with respect to the state vector and with respect to the parameter scaling the variance/covariance matrix of the shocks evaluated at the steady state is zero. This result implies that for any model belonging to the general class considered in this paper, the coefficients on the terms linear and quadratic in the state vector in a second-order expansion of the decision rule are independent of the volatility of the exogenous shocks. In other words, these coefficients must be the same in the stochastic and the deterministic versions of the model. Thus, up to second order, the presence of uncertainty affects only the constant term of the decision rules.

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<sup>1</sup> See [Woodford \(2002\)](#) for a discussion of conditions under which it is correct up to second order to approximate the level of welfare using first-order approximations to the policy function.

<sup>2</sup> [Judd \(1998, pp. 477–480\)](#) obtains this result in the context of a simple one-sector, stochastic, discrete-time growth model. Thus, our theoretical finding can be viewed as a generalization of [Judd's](#) result to a wide class of rational expectations models.

The usefulness of our theoretical results can be illustrated by relating them to recent work on second-order approximation techniques by Collard and Juillard (2001a, b) and Sims (2000). We follow Collard and Juillard closely in notation and methodology. However, an important difference separates our paper from their work. Namely, Collard and Juillard apply a fixed-point algorithm, which they call ‘bias reduction procedure,’ to capture the fact that the policy function depends on the variance of the underlying shocks. Their procedure makes the coefficients of the approximated policy rule that are linear and quadratic in the state vector functions of the size of the volatility of the exogenous shocks. By the main theoretical result of this paper, those coefficients are, up to second order, independent of the standard deviation of the shocks. It follows that the bias reduction procedure of Collard and Juillard is not equivalent to a second-order Taylor approximation to the decision rules.<sup>3</sup>

Sims (2000) also derives a second-order approximation to the policy function for a wide class of discrete-time models. In his derivation, Sims (2000) correctly assumes that the coefficients on the terms linear and quadratic in the state vector do not depend on the volatility of the shock and obtains a second-order approximation to the policy function that is valid only under this assumption. However, he does not provide the proof that this must be the case. Our paper provides this proof in a general setting.

At a practical level, our paper contributes to the existing literature by providing MATLAB code to compute second-order approximations for any rational expectations model whose equilibrium conditions can be written in the general form considered in this paper. We demonstrate the ability of this code to deliver accurate second-order approximations by applying it to a number of example economies.

The first example considered is the standard, one-sector, stochastic growth model. Sims (2000) computes a second-order approximation to this economy, which we are able to replicate. The second example applies our code to the two-country growth model with complete asset markets studied by Kim and Kim (in press). This economy features multiple state variables. Kim and Kim have derived analytically the second-order approximation to the policy function of this economy. We use this example to verify that our code delivers correct answers in a multi-state environment. Finally, we apply our code to the asset-pricing model of Burnside (1998). This example is also analyzed in Collard and Juillard (2001b). Burnside solves this model analytically. Thus, we can derive analytically the second-order approximation to the policy function. This example serves two purposes. First, it gives support to the validity of our code. Second, it allows us to quantify the differences between the Taylor second-order approximation and the bias reduction procedure of Collard and Juillard (2001a, b).

The remainder of the paper is organized as follows. In the next section we present the model. In Section 3 we derive first- and second-order approximations to the policy function. In Section 4 we describe the Matlab computer code designed to implement the second-order approximation to the policy rules. Section 5 closes the paper with applications of the algorithm developed in this paper to three example economies.

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<sup>3</sup> The procedure adopted by Collard and Juillard can be interpreted as having the objective of fitting a second-order polynomial approximation to the policy function for a specific value of the standard deviation of the vector of exogenous shocks strictly greater than zero.

**2. The model**

The set of equilibrium conditions of a wide variety of dynamic general equilibrium models in macroeconomics can be written as

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = 0, \tag{1}$$

where  $E_t$  denotes the mathematical expectations operator conditional on information available at time  $t$ . The vector  $x_t$  of predetermined variables is of size  $n_x \times 1$  and the vector  $y_t$  of non-predetermined variables is of size  $n_y \times 1$ . We define  $n = n_x + n_y$ . The function  $f$  maps  $R^{n_y} \times R^{n_y} \times R^{n_x} \times R^{n_x}$  into  $R^n$ . The state vector  $x_t$  can be partitioned as  $x_t = [x_t^1; x_t^2]'$ . The vector  $x_t^1$  consists of endogenous predetermined state variables and the vector  $x_t^2$  of exogenous state variables. Specifically, we assume that  $x_t^2$  follows the exogenous stochastic process given by

$$x_{t+1}^2 = Ax_t^2 + \tilde{\eta}\sigma\varepsilon_{t+1},$$

where both the vector  $x_t^2$  and the innovation  $\varepsilon_t$  are of order  $n_e \times 1$ . The vector  $\varepsilon_t$  is assumed to have a bounded support and to be independently and identically distributed, with mean zero and variance/covariance matrix  $I$ .<sup>4</sup>

The scalar  $\sigma \geq 0$  and the  $n_e \times n_e$  matrix  $\tilde{\eta}$  are known parameters. All eigenvalues of the matrix  $A$  are assumed to have modulus less than one.<sup>5</sup>

To see how the equilibrium conditions of a familiar model can be expressed in the form given by Eq. (1), consider the simple neoclassical growth model. Its equilibrium conditions are given by

$$c_t^{-\gamma} = \beta E_t c_{t+1}^{-\gamma} [\alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta],$$

$$c_t + k_{t+1} = A_t k_t^\alpha + (1 - \delta)k_t,$$

$$\ln A_{t+1} = \rho \ln A_t + \sigma \varepsilon_{t+1},$$

for all  $t \geq 0$ , given  $k_0$  and  $A_0$ . Let  $y_t = c_t$  and  $x_t = [k_t; \ln A_t]'$ . Then

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = E_t \begin{bmatrix} y_{1t}^{-\gamma} - \beta y_{1t+1}^{-\gamma} [\alpha e^{x_{2t+1}} x_{1t+1}^{\alpha-1} + 1 - \delta] \\ y_{1t} + x_{1t+1} - e^{x_{2t}} x_{1t}^\alpha - (1 - \delta)x_{1t} \\ x_{2t+1} - \rho x_{2t} \end{bmatrix},$$

where  $x_{it}$  and  $y_{it}$  denote, respectively, the  $i$ th element of the vectors  $x_t$  and  $y_t$ .

<sup>4</sup> See Samuelson (1970) and Jin and Judd (2002) for discussions of what might go wrong when the vector of exogenous shocks is allowed to have an unbounded support.

<sup>5</sup> Note that our formulation allows for any number of lags in endogenous and exogenous state variables. Also, it is straightforward to accommodate a more general law of motion for  $x_t^2$  of the form  $x_{t+1}^2 = \Gamma(x_t^2) + \sigma \tilde{\eta} \varepsilon_{t+1}$ , where  $\Gamma$  is a non-linear function satisfying the condition that all eigenvalues of its first derivative evaluated at the non-stochastic steady state lie within the unit circle. Further, the size of the innovation  $\varepsilon_t$  need not equal that of  $x_t^2$ .

We now return to the general case. The solution to the model given in Eq. (1) is of the form:

$$y_t = g(x_t, \sigma), \tag{2}$$

$$x_{t+1} = h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1}, \tag{3}$$

where  $g$  maps  $R^{n_x} \times R^+$  into  $R^{n_y}$  and  $h$  maps  $R^{n_x} \times R^+$  into  $R^{n_x}$ . The matrix  $\eta$  is of order  $n_x \times n_\varepsilon$  and is given by

$$\eta = \begin{bmatrix} \emptyset \\ \tilde{\eta} \end{bmatrix}.$$

We wish to find a second-order approximation of the functions  $g$  and  $h$  around the non-stochastic steady state,  $x_t = \bar{x}$  and  $\sigma = 0$ . We define the non-stochastic steady state as vectors  $(\bar{x}, \bar{y})$  such that

$$f(\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0.$$

It is clear that  $\bar{y} = g(\bar{x}, 0)$  and  $\bar{x} = h(\bar{x}, 0)$ . To see this, note that if  $\sigma = 0$ , then  $E_t f = f$ .

### 3. Approximating the solution

Substituting the proposed solution given by Eqs. (2) and (3) into Eq. (1), we can define

$$\begin{aligned} F(x, \sigma) &\equiv E_t f(g(h(x, \sigma) + \eta \sigma \varepsilon', \sigma), g(x, \sigma), h(x, \sigma) + \eta \sigma \varepsilon', x) \\ &= 0. \end{aligned} \tag{4}$$

Here we are dropping time subscripts. We use a prime to indicate variables dated in period  $t + 1$ .

Because  $F(x, \sigma)$  must be equal to zero for any possible values of  $x$  and  $\sigma$ , it must be the case that the derivatives of any order of  $F$  must also be equal to zero. Formally,

$$F_{x^k \sigma^j}(x, \sigma) = 0 \quad \forall x, \sigma, j, k, \tag{5}$$

where  $F_{x^k \sigma^j}(x, \sigma)$  denotes the derivative of  $F$  with respect to  $x$  taken  $k$  times and with respect to  $\sigma$  taken  $j$  times.

#### 3.1. First-order approximation

We are looking for approximations to  $g$  and  $h$  around the point  $(x, \sigma) = (\bar{x}, 0)$  of the form

$$g(x, \sigma) = g(\bar{x}, 0) + g_x(\bar{x}, 0)(x - \bar{x}) + g_\sigma(\bar{x}, 0)\sigma,$$

$$h(x, \sigma) = h(\bar{x}, 0) + h_x(\bar{x}, 0)(x - \bar{x}) + h_\sigma(\bar{x}, 0)\sigma.$$

As explained earlier,

$$g(\bar{x}, 0) = \bar{y}$$

and

$$h(\bar{x}, 0) = \bar{x}.$$

The remaining unknown coefficients of the first-order approximation to  $g$  and  $h$  are identified by using the fact that, by Eq. (5), it must be the case that:

$$F_x(\bar{x}, 0) = 0$$

and

$$F_\sigma(\bar{x}, 0) = 0.$$

Thus, using the first of these two expressions,  $g_x$  and  $h_x$  can be found as the solution to the system

$$[F_x(\bar{x}, 0)]_j^i = [f_{y'}]_\alpha^i [g_x]_\beta^\alpha [h_x]_j^\beta + [f_y]_\alpha^i [g_x]_\beta^\alpha + [f_{x'}]_\beta^i [h_x]_j^\beta + [f_x]_\alpha^i = 0; \quad i = 1, \dots, n; \quad j, \beta = 1, \dots, n_x; \quad \alpha = 1, \dots, n_y$$

Here we are using the notation suggested by Collard and Juillard (2001a). So, for example,  $[f_{y'}]_\alpha^i$  is the  $(i, \alpha)$  element of the derivative of  $f$  with respect to  $y'$ . The derivative of  $f$  with respect to  $y'$  is an  $n \times n_y$  matrix. Therefore,  $[f_{y'}]_\alpha^i$  is the element of this matrix located at the intersection of the  $i$ th row and  $\alpha$ th column. Also, for example,  $[f_{y'}]_\alpha^i [g_x]_\beta^\alpha [h_x]_j^\beta = \sum_{\alpha=1}^{n_y} \sum_{\beta=1}^{n_x} (\partial f^i / \partial y'^\alpha) (\partial g^\alpha / \partial x^\beta) (\partial h^\beta / \partial x^j)$ .

Note that the derivatives of  $f$  evaluated at  $(y', y, x', x) = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$  are known. The above expression represents a system of  $n \times n_x$  quadratic equations in the  $n \times n_x$  unknowns given by the elements of  $g_x$  and  $h_x$ .<sup>6</sup>

Similarly,  $g_\sigma$  and  $h_\sigma$  are identified as the solution to the following  $n$  equations:

$$[F_\sigma(\bar{x}, 0)]^i = E_t \{ [f_{y'}]_\alpha^i [g_x]_\beta^\alpha [h_\sigma]^\beta + [f_{y'}]_\alpha^i [g_x]_\beta^\alpha [\eta]_\phi^\beta [\varepsilon']^\phi + [f_{y'}]_\alpha^i [g_\sigma]^\alpha + [f_y]_\alpha^i [g_\sigma]^\alpha + [f_{x'}]_\beta^i [h_\sigma]^\beta + [f_{x'}]_\beta^i [\eta]_\phi^\beta [\varepsilon']^\phi \} = [f_{y'}]_\alpha^i [g_x]_\beta^\alpha [h_\sigma]^\beta + [f_{y'}]_\alpha^i [g_\sigma]^\alpha + [f_y]_\alpha^i [g_\sigma]^\alpha + [f_{x'}]_\beta^i [h_\sigma]^\beta = 0; \quad i = 1, \dots, n; \quad \alpha = 1, \dots, n_y; \quad \beta = 1, \dots, n_x; \quad \phi = 1, \dots, n_\varepsilon. \tag{6}$$

<sup>6</sup> A number of authors have developed algorithms for finding solutions to the above equation associated with non-explosive paths for the state and control variables (e.g., Blanchard and Kahn, 1985; Sims, 2002; and Klein (2000)). In the numerical applications presented at the end of the paper we used Klein's package to solve for  $g_x$  and  $h_x$ .

Note that this equation is linear and homogeneous in  $g_\sigma$  and  $h_\sigma$ . Thus, if a unique solution exists, we have that

$$h_\sigma = 0$$

and

$$g_\sigma = 0.$$

These two expressions represent our first main theoretical result. They show that in general, up to first order, one need not correct the constant term of the approximation to the policy function for the size of the variance of the shocks. This result implies that in a first-order approximation the expected values of  $x_t$  and  $y_t$  are equal to their non-stochastic steady-state values  $\bar{x}$  and  $\bar{y}$ .

### 3.2. Second-order approximation

The second-order approximations to  $g$  and  $h$  around the point  $(x, \sigma) = (\bar{x}, 0)$  are of the form

$$\begin{aligned} [g(x, \sigma)]^i &= [g(\bar{x}, 0)]^i + [g_x(\bar{x}, 0)]^i_a [(x - \bar{x})]_a + [g_\sigma(\bar{x}, 0)]^i [\sigma] \\ &\quad + \frac{1}{2} [g_{xx}(\bar{x}, 0)]^i_{ab} [(x - \bar{x})]_a [(x - \bar{x})]_b \\ &\quad + \frac{1}{2} [g_{x\sigma}(\bar{x}, 0)]^i_a [(x - \bar{x})]_a [\sigma] \\ &\quad + \frac{1}{2} [g_{\sigma x}(\bar{x}, 0)]^i_a [(x - \bar{x})]_a [\sigma] \\ &\quad + \frac{1}{2} [g_{\sigma\sigma}(\bar{x}, 0)]^i [\sigma] [\sigma] \end{aligned}$$

$$\begin{aligned} [h(x, \sigma)]^j &= [h(\bar{x}, 0)]^j + [h_x(\bar{x}, 0)]^j_a [(x - \bar{x})]_a + [h_\sigma(\bar{x}, 0)]^j [\sigma] \\ &\quad + \frac{1}{2} [h_{xx}(\bar{x}, 0)]^j_{ab} [(x - \bar{x})]_a [(x - \bar{x})]_b \\ &\quad + \frac{1}{2} [h_{x\sigma}(\bar{x}, 0)]^j_a [(x - \bar{x})]_a [\sigma] \\ &\quad + \frac{1}{2} [h_{\sigma x}(\bar{x}, 0)]^j_a [(x - \bar{x})]_a [\sigma] \\ &\quad + \frac{1}{2} [h_{\sigma\sigma}(\bar{x}, 0)]^j [\sigma] [\sigma], \end{aligned}$$

where  $i = 1, \dots, n_y$ ,  $a, b = 1, \dots, n_x$ , and  $j = 1, \dots, n_x$ . The unknowns of this expansion are  $[g_{xx}]^i_{ab}$ ,  $[g_{x\sigma}]^i_a$ ,  $[g_{\sigma x}]^i_a$ ,  $[g_{\sigma\sigma}]^i$ ,  $[h_{xx}]^j_{ab}$ ,  $[h_{x\sigma}]^j_a$ ,  $[h_{\sigma x}]^j_a$ ,  $[h_{\sigma\sigma}]^j$ , where we have omitted the argument  $(\bar{x}, 0)$ . These coefficients can be identified by taking the derivative of  $F(x, \sigma)$  with respect to  $x$  and  $\sigma$  twice and evaluating them at  $(x, \sigma) = (\bar{x}, 0)$ . By the

arguments provided earlier, these derivatives must be zero. Specifically, we use  $F_{xx}(\bar{x}, 0)$  to identify  $g_{xx}(\bar{x}, 0)$  and  $h_{xx}(\bar{x}, 0)$ . That is,<sup>7</sup>

$$\begin{aligned}
 [F_{xx}(\bar{x}, 0)]_{jk}^i &= ([f_{y'y'}]_{\alpha\gamma}^i [g_x]_{\delta}^{\gamma} [h_x]_k^{\delta} + [f_{y'y'}]_{\alpha\gamma}^i [g_x]_k^{\gamma}) \\
 &\quad + [f_{y'x'}]_{\alpha\delta}^i [h_x]_k^{\delta} + [f_{y'x'}]_{\alpha k}^i [g_x]_{\beta}^{\alpha} [h_x]_j^{\beta} \\
 &\quad + [f_{y'}]_{\alpha}^i [g_{xx}]_{\beta\delta}^{\alpha} [h_x]_k^{\delta} [h_x]_j^{\beta} \\
 &\quad + [f_{y'}]_{\alpha}^i [g_x]_{\beta}^{\alpha} [h_{xx}]_{jk}^{\beta} \\
 &\quad + ([f_{yy'}]_{\alpha\gamma}^i [g_x]_{\delta}^{\gamma} [h_x]_k^{\delta} + [f_{yy'}]_{\alpha\gamma}^i [g_x]_k^{\gamma} + [f_{yx'}]_{\alpha\delta}^i [h_x]_k^{\delta} + [f_{yx'}]_{\alpha k}^i [g_x]_j^{\alpha}) \\
 &\quad + [f_{y'}]_{\alpha}^i [g_{xx}]_{jk}^{\alpha} \\
 &\quad + ([f_{x'y'}]_{\beta\gamma}^i [g_x]_{\delta}^{\gamma} [h_x]_k^{\delta} + [f_{x'y'}]_{\beta\gamma}^i [g_x]_k^{\gamma} + [f_{x'x'}]_{\beta\delta}^i [h_x]_k^{\delta} + [f_{x'x'}]_{\beta k}^i) [h_x]_j^{\beta} \\
 &\quad + [f_{x'}]_{\beta}^i [h_{xx}]_{jk}^{\beta} \\
 &\quad + [f_{x'y'}]_{j\gamma}^i [g_x]_{\delta}^{\gamma} [h_x]_k^{\delta} + [f_{x'y'}]_{j\gamma}^i [g_x]_k^{\gamma} + [f_{xx'}]_{j\delta}^i [h_x]_k^{\delta} + [f_{xx}]_{jk}^i \\
 &= 0; \quad i = 1, \dots, n, \quad j, k, \beta, \delta = 1, \dots, n_x; \quad \alpha, \gamma = 1, \dots, n_y.
 \end{aligned}$$

Since we know the derivatives of  $f$  as well as the first derivatives of  $g$  and  $h$  evaluated at  $(y', y, x', x) = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$ , it follows that the above expression represents a system of  $n \times n_x \times n_x$  linear equations in the  $n \times n_x \times n_x$  unknowns given by the elements of  $g_{xx}$  and  $h_{xx}$ .

Similarly,  $g_{\sigma\sigma}$  and  $h_{\sigma\sigma}$  can be obtained by solving the linear system  $F_{\sigma\sigma}(\bar{x}, 0) = 0$ . More explicitly,

$$\begin{aligned}
 [F_{\sigma\sigma}(\bar{x}, 0)]^i &= [f_{y'}]_{\alpha}^i [g_x]_{\beta}^{\alpha} [h_{\sigma\sigma}]^{\beta} \\
 &\quad + [f_{y'y'}]_{\alpha\gamma}^i [g_x]_{\delta}^{\gamma} [\eta]_{\xi}^{\delta} [g_x]_{\beta}^{\alpha} [\eta]_{\phi}^{\beta} [I]_{\xi}^{\phi} \\
 &\quad + [f_{y'x'}]_{\alpha\delta}^i [\eta]_{\xi}^{\delta} [g_x]_{\beta}^{\alpha} [\eta]_{\phi}^{\beta} [I]_{\xi}^{\phi} \\
 &\quad + [f_{y'}]_{\alpha}^i [g_{xx}]_{\beta\delta}^{\alpha} [\eta]_{\xi}^{\delta} [\eta]_{\phi}^{\beta} [I]_{\xi}^{\phi} \\
 &\quad + [f_{y'}]_{\alpha}^i [g_{\sigma\sigma}]^{\alpha} \\
 &\quad + [f_{y'}]_{\alpha}^i [g_{\sigma\sigma}]^{\alpha} \\
 &\quad + [f_{x'}]_{\beta}^i [h_{\sigma\sigma}]^{\beta}
 \end{aligned}$$

<sup>7</sup> At this point, an additional word about notation is in order. Take for example the expression  $[f_{y'y'}]_{\alpha\gamma}^i$ . Note that  $f_{y'y'}$  is a three dimensional array with  $n$  rows,  $n_y$  columns, and  $n_y$  pages. Then  $[f_{y'y'}]_{\alpha\gamma}^i$  denotes the element of  $f_{y'y'}$  located at the intersection of row  $i$ , column  $\alpha$  and page  $\gamma$ .



$$\begin{aligned}
 & + [f_{x'y'}]_{\beta\gamma}^i [g_x]_{\delta}^{\gamma} [\eta]_{\xi}^{\delta} [\eta]_{\phi}^{\beta} [I]_{\xi}^{\phi} \\
 & + [f_{x'x'}]_{\beta\delta}^i [\eta]_{\xi}^{\delta} [\eta]_{\phi}^{\beta} [I]_{\xi}^{\phi} \\
 = & 0; \quad i = 1, \dots, n; \quad \alpha, \gamma = 1, \dots, n_y; \quad \beta, \delta = 1, \dots, n_x; \\
 & \phi, \xi = 1, \dots, n_e.
 \end{aligned} \tag{7}$$

This is a system of  $n$  linear equations in the  $n$  unknowns given by the elements of  $g_{\sigma\sigma}$  and  $h_{\sigma\sigma}$ .

Finally, we show that the cross derivatives  $g_{x\sigma}$  and  $h_{x\sigma}$  are equal to zero when evaluated at  $(\bar{x}, 0)$ . We write the system  $F_{\sigma x}(\bar{x}, 0) = 0$  taking into account that all terms containing either  $g_{\sigma}$  or  $h_{\sigma}$  are zero at  $(\bar{x}, 0)$ . Then we have,

$$\begin{aligned}
 [F_{\sigma x}(\bar{x}, 0)]_j^i & = [f_{y'}]_{\alpha}^i [g_x]_{\beta}^{\alpha} [h_{\sigma x}]_j^{\beta} + [f_{y'}]_{\alpha}^i [g_{\sigma x}]_{\gamma}^{\alpha} [h_x]_j^{\gamma} + [f_{y'}]_{\alpha}^i [g_{\sigma x}]_j^{\alpha} + [f_{x'}]_{\beta}^i [h_{\sigma x}]_j^{\beta} \\
 = & 0; \quad i = 1, \dots, n; \quad \alpha = 1, \dots, n_y; \quad \beta, \gamma, j = 1, \dots, n_x.
 \end{aligned} \tag{8}$$

This is a system of  $n \times n_x$  equations in the  $n \times n_x$  unknowns given by the elements of  $g_{\sigma x}$  and  $h_{\sigma x}$ . But clearly, the system is homogeneous in the unknowns. Thus, if a unique solution exists, it is given by

$$g_{\sigma x} = 0$$

and

$$h_{\sigma x} = 0.$$

These equations represent our second main theoretical result. They show that in general, up to second-order, the coefficients of the policy function on the terms that are linear in the state vector do not depend on the size of the variance of the underlying shocks.<sup>8</sup> We summarize our two main theoretical results in the following theorem:

**Theorem 1.** *Consider the model given by Eq. (1) and its solution given by the policy functions (2) and (3). Then*

$$g_{\sigma}(\bar{x}, 0) = 0,$$

$$h_{\sigma}(\bar{x}, 0) = 0,$$

$$g_{x\sigma}(\bar{x}, 0) = 0, \quad \text{and}$$

$$h_{x\sigma}(\bar{x}, 0) = 0.$$

Theorem 1 shows that the second-order approximation to the policy function of a stochastic model belonging to the general class given in Eq. (1) differs from that of its non-stochastic counterpart only in a constant term given by  $\frac{1}{2}g_{\sigma\sigma}\sigma^2$  for the control

<sup>8</sup> Chen and Zadrozny (2003) obtain similar results in the context of a linear-quadratic exponential Gaussian optimal control problem. Also, upon receiving a draft of our paper, Ken Judd communicated to us that in work in progress he came across results similar to those we obtain in this section.

vector  $y_t$  and by the first  $n_x - n_e$  elements of  $\frac{1}{2}h_{\sigma\sigma}\sigma^2$  for the endogenous state vector  $x_t^1$ . Therefore, any second-order expansion of the policy function of a stochastic problem whose linear and quadratic coefficients do not coincide with those of the non-stochastic version of the model does not represent a second-order Taylor expansion. This is the case, for instance, with the approximation resulting from the bias reduction procedure of Collard and Juillard (2001a, b).<sup>9</sup> In Section 5.2, we quantify the differences between a second-order Taylor expansion and the bias reduction procedure for a particular model economy.

### 3.3. Higher-order approximations

It is straightforward to apply the method described thus far to finding higher-order approximations to the policy function. For example, given the first- and second-order terms of the Taylor expansions of  $h$  and  $g$ , the third-order terms can be identified by solving a linear system of equations. More generally, one can construct sequentially the  $n$ th-order approximation of the policy function by solving a linear system of equations whose (known) coefficients are the lower-order terms and the derivatives up to order  $n$  of  $f$  evaluated at  $(y', y, x', x) = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$  (see also Collard and Juillard, 2001a; and Judd, 1998).<sup>10</sup>

## 4. Matlab codes

We prepared a set of Matlab codes that implements the second-order approximation developed above. The only inputs that the user needs to provide are the set of equilibrium conditions (the function  $f(y_{t+1}, y_t, x_{t+1}, x_t)$ ) and the non-stochastic steady-state values of  $x_t$  and  $y_t$  given by  $\bar{x}$  and  $\bar{y}$ . In our programs, much work and possibilities for errors are eliminated by using the MATLAB toolbox Symbolic Math to compute analytically the first and second derivatives of the function  $f$ .

The programs are publicly available on the world wide web.<sup>11</sup> The program `gx_hx.m` computes the matrices  $g_x$  and  $h_x$ . The inputs to the program are the first derivatives of  $f$  evaluated at the steady state. That is,  $f_y, f_x, f_{y'}$ , and  $f_{x'}$ . This step amounts to obtaining a first-order approximation to the policy functions. A number of packages are available for this purpose. We use the one prepared by Paul Klein of the University

<sup>9</sup> Collard and Juillard (2001b) maintain that the presence of stochastic shocks will in general affect the coefficients on the terms that are linear or quadratic in  $x_t$  in a second-order approximation. Specifically, on p. 984 they state that “[i]t should be clear to the reader that, as higher-order moments will be taken into account, values for  $f_0$  and  $f_1$  [in our notation,  $f_1$  corresponds to the coefficient on  $x_t - \bar{x}$  in the second-order expansion] will be affected. More particularly, both of them will now depend on volatilities.”

<sup>10</sup> We conjecture the following generalization of Theorem 1: The derivatives of the policy function with respect to  $x$  any number of times and with respect to  $\sigma$  an odd number of times evaluated at the non-stochastic steady state are zero. We base this conjecture on numerical results obtained by Jesús Fernández-Villaverde of the University of Pennsylvania who extended our method to accommodate higher order approximations using Mathematica and applied it to a number of variations of the neoclassical growth model.

<sup>11</sup> The URL is [http://www.econ.upenn.edu/~uribe/2nd\\_order.htm](http://www.econ.upenn.edu/~uribe/2nd_order.htm).

of Western Ontario, which consists of the three programs `solab.m`, `qzswitch.m`, and `reorder.m`.<sup>12</sup>

The program `gxx_hxx.m` computes the arrays  $g_{xx}$  and  $h_{xx}$ . The inputs to the program are the first and second derivatives of  $f$  evaluated at the steady state and the matrices  $g_x$  and  $h_x$  produced by `gx_hx.m`.

The program `gss_hss.m` computes the arrays  $g_{\sigma\sigma}$  and  $h_{\sigma\sigma}$ . The inputs to the program are the first and second derivatives of  $f$  evaluated at the steady state, the matrices  $g_x$  and  $h_x$  produced by the program `gx_hx.m`, the array  $g_{xx}$  produced by the program `gxx_hxx.m`, and the matrix  $\eta$ .

#### 4.1. Computing the derivatives of $f$

Computing the derivatives of  $f$ , particularly the second derivatives, can be a daunting task if the model is large. We approach this problem as follows. The MATLAB Toolbox Symbolic Math can handle analytical derivatives. We wrote programs that compute the analytical derivatives of  $f$  and evaluate them at the steady state. The program `anal_deriv.m` computes the analytical derivatives of  $f$  and the program `num_eval.m` evaluates the analytical derivatives of  $f$  at the steady state.

#### 4.2. Examples

To illustrate the use of the programs described thus far, we posted on the website given above the programs needed to obtain the second-order approximation to the decision rules of the three model economies studied in Section 5. For example, to obtain the second-order approximation to the policy functions of the neoclassical growth model discussed in Sections 2 and 5, run the program `neoclassical_model_run.m`. The output of this program are the matrices  $g_x$  and  $h_x$  and the arrays  $g_{xx}$ ,  $h_{xx}$ ,  $g_{\sigma\sigma}$  and  $h_{\sigma\sigma}$ . This program calls the program `neoclassical_model.m`, which produces the first- and second derivatives of  $f$ . More generally, `neoclassical_model.m` illustrates how to write down analytically the equations of a model belonging to the class given in Eq. (1) using the MATLAB Toolbox Symbolic Math.

### 5. Applications

In this section, we apply the second-order approximation method developed above and the computer code that implements it to solve numerically for the equilibrium dynamics of a number of models. These models were chosen because they are particularly well suited for evaluating the ability of the proposed algorithm to arrive at the correct second-order approximation to the decision rule. We begin with the one-sector neoclassical growth model for which Sims (2000) has computed a second-order approximation. This is an economy with one endogenous predetermined state and one control variable. We then consider a two-country growth model with complete asset

<sup>12</sup> For a description of the technique used in this package, see Klein (2000).

markets. In this case there are two endogenous predetermined variables and one control. For this economy, Kim and Kim (in press) derive analytically the second-order approximation to the policy function in the case that the underlying shocks are iid. We close the section with an examination of an asset pricing model that has a closed form solution due to Burnside (1998). Recently, Collard and Juillard (2001b) have used this model as a benchmark to evaluate the accuracy of their perturbation method, which incorporates an iterative procedure to capture the effects of the presence of uncertainty on the coefficients of the second-order expansion. As we show above, this procedure introduces a discrepancy in the second-order approximation with respect to the second-order Taylor expansion. We present parameterization under which these differences are quantitatively large.

### 5.1. Example 1: The neoclassical growth model

Consider the simple neoclassical model, described in Section 2. We calibrate the model by setting  $\beta = 0.95$ ,  $\delta = 1$ ,  $\alpha = 0.3$ ,  $\rho = 0$ , and  $\gamma = 2$ . We choose these parameter values to facilitate comparison with the results obtained by applying Sims's (2000) method.<sup>13</sup> Here we are interested in a quadratic approximation to the policy function around the natural logarithm of the steady state. Thus, unlike in Section 2, we now define:

$$x_t = \begin{bmatrix} \ln k_t \\ \ln A_t \end{bmatrix}$$

and

$$y_t = \ln c_t.$$

Then the non-stochastic steady-state values of  $y_t$  and  $x_t$  are, respectively:

$$\bar{y} = -0.8734.$$

and

$$\bar{x} = \begin{bmatrix} -1.7932 \\ 0 \end{bmatrix}.$$

The coefficients of the linear terms are:

$$g_x = [0.2525 \quad 0.8417]$$

and

$$h_x = \begin{bmatrix} 0.4191 & 1.3970 \\ 0.0000 & 0.0000 \end{bmatrix}.$$

<sup>13</sup> See the MATLAB script sessionEG.m in Sims's website (<http://eco-072399b.princeton.edu/yftp/gensys2/GrowthEG>)

The coefficients of the quadratic terms are given by:

$$g_{xx}(:, :, 1) = [-0.0051 \quad -0.0171],$$

$$g_{xx}(:, :, 2) = [-0.0171 \quad -0.0569]$$

and

$$h_{xx}(:, :, 1) = \begin{bmatrix} -0.0070 & -0.0233 \\ 0 & 0 \end{bmatrix},$$

$$h_{xx}(:, :, 2) = \begin{bmatrix} -0.0233 & -0.0778 \\ 0 & 0 \end{bmatrix}.$$

Finally, the coefficients of the quadratic terms in  $\sigma$  are:

$$g_{\sigma\sigma} = -0.1921$$

and

$$h_{\sigma\sigma} = \begin{bmatrix} 0.4820 \\ 0 \end{bmatrix}.$$

A more familiar representation is given by the evolution of the original variables. Let

$$\hat{c}_t \equiv \ln(c_t/\bar{c})$$

and

$$\hat{k}_t \equiv \ln(k_t/\bar{k}).$$

Then, the laws of motion of these two variables are given by

$$\hat{c}_t = 0.2525\hat{k}_t + 0.8417\hat{A}_t + \frac{1}{2}[-0.0051\hat{k}_t^2 - 0.0341\hat{k}_t\hat{A}_t - 0.0569\hat{A}_t^2 - 0.1921\sigma^2]$$

and

$$\begin{aligned} \hat{k}_{t+1} &= 0.4191\hat{k}_t + 1.3970\hat{A}_t \\ &+ \frac{1}{2}[-0.0070\hat{k}_t^2 - 0.0467\hat{k}_t\hat{A}_t - 0.0778\hat{A}_t^2 + 0.4820\sigma^2]. \end{aligned}$$

It can be verified that these numbers coincide with those obtained by Sims (2000).

## 5.2. Example 2: A two-country neoclassical model with complete asset markets

The following 2-country international real business cycle model with complete asset markets is taken from Kim and Kim (in press). The competitive equilibrium real allocations associated with this economy can be obtained by solving the first-best problem.

The planner’s objective function is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_{1t}^{1-\gamma} - 1}{1-\gamma} + \frac{C_{2t}^{1-\gamma} - 1}{1-\gamma} \right],$$

where  $C_{it}$ ,  $i = 1, 2$ , denotes consumption of the representative household of country  $i$  in period  $t$ . The planner maximizes this utility function subject to the budget constraint

$$C_{1t} + C_{2t} + K_{1t+1} - (1 - \delta)K_{1t} + K_{2t+1} - (1 - \delta)K_{2t} = A_{1t}K_{1t}^\alpha + A_{2t}K_{2t}^\alpha,$$

where  $K_{it}$  denotes the stock of physical capital in country  $i$  and  $A_{it}$  is an exogenous technology shock whose law of motion is given by

$$\ln A_{it} = \rho_i \ln A_{it-1} + \sigma \varepsilon_{it}, \quad i = 1, 2,$$

where  $\varepsilon_{it} \sim NIID(0, 1)$  and  $\rho_i \in (-1, 1)$ . The optimality conditions associated with this problem are the above period-by-period budget constraint and

$$C_{1t} = C_{2t},$$

$$C_{1t}^{-\gamma} = \beta E_t C_{1t+1}^{-\gamma} [\alpha A_{1t+1} K_{1t+1}^{\alpha-1} + (1 - \delta)],$$

$$C_{1t}^{-\gamma} = \beta E_t C_{1t+1}^{-\gamma} [\alpha A_{2t+1} K_{2t+1}^{\alpha-1} + (1 - \delta)].$$

We use the following parameter values:  $\gamma = 2$ ;  $\delta = 0.1$ ;  $\alpha = 0.3$ ;  $\rho = 0$ ; and  $\beta = 0.95$ .

Given this parameterization, the second-order approximation to the policy function is given by:

$$\hat{K}_{1t+1} = [0.4440 \quad 0.4440 \quad 0.2146 \quad 0.2146] \begin{bmatrix} \hat{K}_{1t} \\ \hat{K}_{2t} \\ \hat{A}_{1t} \\ \hat{A}_{2t} \end{bmatrix} + \frac{1}{2} [\hat{K}_{1t} \quad \hat{K}_{2t} \quad \hat{A}_{1t} \quad \hat{A}_{2t}] \times \begin{bmatrix} 0.22 & -0.18 & -0.023 & -0.088 \\ -0.18 & 0.22 & -0.088 & -0.023 \\ -0.023 & -0.088 & 0.17 & -0.042 \\ -0.088 & -0.023 & -0.042 & 0.17 \end{bmatrix} \begin{bmatrix} \hat{K}_{1t} \\ \hat{K}_{2t} \\ \hat{A}_{1t} \\ \hat{A}_{2t} \end{bmatrix} - \frac{1}{2} 0.166 \sigma^2,$$

$$\hat{K}_{2t+1} = \hat{K}_{1t+1},$$

and

$$\begin{aligned} \hat{C}_{1t} = & [0.2 \quad 0.2 \quad 0.097 \quad 0.097] \begin{bmatrix} \hat{K}_{1t} \\ \hat{K}_{2t} \\ \hat{A}_{1t} \\ \hat{A}_{2t} \end{bmatrix} \\ & + \frac{1}{2} [\hat{K}_{1t} \quad \hat{K}_{2t} \quad \hat{A}_{1t} \quad \hat{A}_{2t}] \\ & \times \begin{bmatrix} 0.1 & -0.08 & -0.0093 & -0.038 \\ -0.08 & 0.1 & -0.038 & -0.0093 \\ -0.0093 & -0.038 & 0.079 & -0.019 \\ -0.038 & -0.0093 & -0.019 & 0.079 \end{bmatrix} \begin{bmatrix} \hat{K}_{1t} \\ \hat{K}_{2t} \\ \hat{A}_{1t} \\ \hat{A}_{2t} \end{bmatrix} \\ & + \frac{1}{2} 0.406\sigma^2 \\ \hat{C}_{2t} = & \hat{C}_{1t}, \end{aligned}$$

where a hat over a variable denotes the log-deviation from its steady state. The non-stochastic steady state is given by  $[K_i; A_i; C_i] = [2.6257; 1; 1.0733]$ . Kim and Kim (in press) derive analytically a second-order approximation to the decision rule of the model considered here. Our numerical results match those implied by Kim and Kim’s analytical second-order approximation, as can be checked by running the program `kim_run.m`.

### 5.3. Example 3: An asset pricing model

Consider the following endowment economy analyzed by Burnside (1998) and Collard and Juillard (2001b). The representative agent maximizes the lifetime utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^\theta}{\theta},$$

subject to

$$p_t e_{t+1} + C_t = p_t e_t + d_t e_t$$

and a borrowing limit that prevents agents from engaging in Ponzi games. In the above expressions,  $C_t$  denotes consumption,  $p_t$  the relative price of trees in terms of consumption goods,  $e_t$  the number of trees owned by the representative household at the beginning of period  $t$ , and  $d_t$  the dividends per tree in period  $t$ . Dividends are

assumed to follow an exogenous stochastic process given by

$$d_{t+1} = e^{x_{t+1}} d_t,$$

where  $e^{x_t}$  denotes the gross growth rate of dividends. The natural logarithm of the gross growth rate of dividends,  $x_t$ , is assumed to follow an exogenous AR(1) process

$$x_{t+1} = (1 - \rho)\bar{x} + \rho x_t + \sigma \eta \varepsilon_{t+1},$$

and  $\varepsilon_t \sim NIID(0, 1)$ .<sup>14</sup>

The optimality conditions associated with the household’s problem are the above budget constraint, the borrowing limit, and

$$p_t C_t^{\theta-1} = \beta E_t C_{t+1}^{\theta-1} (p_{t+1} + d_{t+1}).$$

In equilibrium we have that  $C_t = d_t$  and  $e_t = 1$ . Defining the price-dividend ratio as  $y_t = p_t/d_t$  yields the equilibrium condition

$$y_t = \beta E_t \{ e^{\theta x_{t+1}} [1 + y_{t+1}] \}.$$

Burnside (1998) shows that the non-explosive solution to this equation is of the form

$$y_t \equiv g(x_t, \sigma) = \sum_{i=1}^{\infty} \beta^i e^{a_i + b_i(x_t - \bar{x})}, \tag{9}$$

where

$$a_i = \theta \bar{x} i + \frac{\theta^2 \sigma^2 \eta^2}{2(1 - \rho)^2} \left[ i - \frac{2\rho(1 - \rho^i)}{1 - \rho} + \frac{\rho^2(1 - \rho^{2i})}{1 - \rho^2} \right]$$

and

$$b_i = \frac{\theta \rho(1 - \rho^i)}{1 - \rho}.$$

It is immediate to see that  $g_{\sigma}(\bar{x}, 0) = g_{x\sigma}(\bar{x}, 0) = 0$ , in line with Theorem 1. Collard and Juillard (2001b) present an algorithm to compute a second-order approximation to the above policy function. Their method appends to a deterministic perturbation method a fixed-point algorithm involving an iterative procedure. This procedure introduces a dependence of the coefficients of the linear and quadratic terms of the expansion of the policy function on the volatility of the underlying shocks. Theorem 1 shows that in a second-order expansion the coefficients on the terms linear and quadratic in the state are independent of the volatility of the exogenous shocks. It follows that the fixed-point algorithm proposed by Collard and Juillard (2001b) yields a second-order approximation that differs from the Taylor second-order approximation.

A second-order approximation of (9) around  $x_t = \bar{x}$  and  $\sigma = 0$  yields

$$y_t \approx g(\bar{x}, 0) + g_x(x_t - \bar{x}) + \frac{1}{2} g_{xx}(x_t - \bar{x})^2 + \frac{1}{2} g_{\sigma\sigma} \sigma^2, \tag{10}$$

<sup>14</sup> To make the notation compatible with that used in previous sections, we call  $\sigma \eta$  what Burnside calls  $\sigma$ .



Table 1  
Second-order approximation to the policy function of the asset-pricing model

	$f_0$	$f_1$	$f_2$
<i>Benchmark calibration:</i>			
2nd-order approx.	12.48	2.27	0.42
Fixed-point algorithm	12.48	2.30	0.43
<i>High curvature, <math>\theta = -10</math>:</i>			
2nd-order approx.	4.79	4.83	6.07
Fixed-point algorithm	5.00	5.97	7.50
<i>High Persistence, <math>\rho = 0.9</math>:</i>			
2nd-order approx.	22.02	-99.07	976.84
Fixed-point algorithm	14.50	-115.40	1137.81

Note: The benchmark calibration is  $\beta = 0.95$ ,  $\theta = -1.5$ ,  $\rho = -0.139$ ,  $\bar{x} = 0.0179$ ,  $\sigma = 1$  and  $\eta = 0.0348$ . The rows labeled '2nd-order approx.' are obtained either by evaluating Eq. (10) or by running the program `asset_run.m`. The rows labeled 'Fixed-Point Algorithm' are taken from Collard and Juillard (2001b), Table 2.

where

$$g_x = \frac{\theta \rho \beta e^{\theta \bar{x}}}{(1 - \beta e^{\theta \bar{x}})(1 - \beta e^{\theta \bar{x}} \rho)},$$

$$g_{xx} = \left( \frac{\rho \theta}{1 - \rho} \right)^2 \left[ \frac{\beta e^{\theta \bar{x}}}{1 - \beta e^{\theta \bar{x}}} - \frac{2\beta e^{\theta \bar{x}} \rho}{1 - \beta e^{\theta \bar{x}} \rho} + \frac{\beta e^{\theta \bar{x}} \rho^2}{1 - \beta e^{\theta \bar{x}} \rho^2} \right],$$

and

$$g_{\sigma\sigma} = \left( \frac{\theta \eta}{1 - \rho} \right)^2 \left[ \frac{\beta e^{\theta \bar{x}}}{(1 - \beta e^{\theta \bar{x}})^2} + \left( \frac{\rho^2}{1 - \rho^2} - \frac{2\rho}{1 - \rho} \right) \right. \\ \left. \times \frac{\beta e^{\theta \bar{x}}}{1 - \beta e^{\theta \bar{x}}} + \frac{2\rho^2}{1 - \rho} \frac{\beta e^{\theta \bar{x}}}{1 - \beta e^{\theta \bar{x}} \rho} - \frac{\rho^4}{1 - \rho^2} \frac{\beta e^{\theta \bar{x}}}{1 - \beta e^{\theta \bar{x}} \rho^2} \right].$$

We follow Burnside (1998) and Collard and Juillard (2001b) and use the calibration  $\beta = 0.95$ ,  $\theta = -1.5$ ,  $\rho = -0.139$ ,  $\bar{x} = 0.0179$ , and  $\eta = 0.0348$ . Then, evaluating the above expressions we obtain

$$y_t \approx 12.30 + 2.27(x_t - \bar{x}) + \frac{1}{2}0.42(x_t - \bar{x})^2 + \frac{1}{2}0.35\sigma^2.$$

This is precisely the equation one obtains using the perturbation algorithm developed in this paper, as can be verified by running the program `asset_run.m`.

Collard and Juillard (2001b) express the second-order approximation to  $y_t$  as

$$y_t = f_0 + f_1(x_t - \bar{x}) + \frac{1}{2}f_2(x_t - \bar{x})^2.$$

Relating their notation to ours, we have  $f_0 = g(\bar{x}, 0) + \frac{1}{2}g_{\sigma\sigma}\sigma^2$ ,  $f_1 = g_x$ , and  $f_2 = g_{xx}$ . In their Table 2, Collard and Juillard (2001b) report the numerical values for  $f_i$  ( $i = 0, 1, 2$ ) for

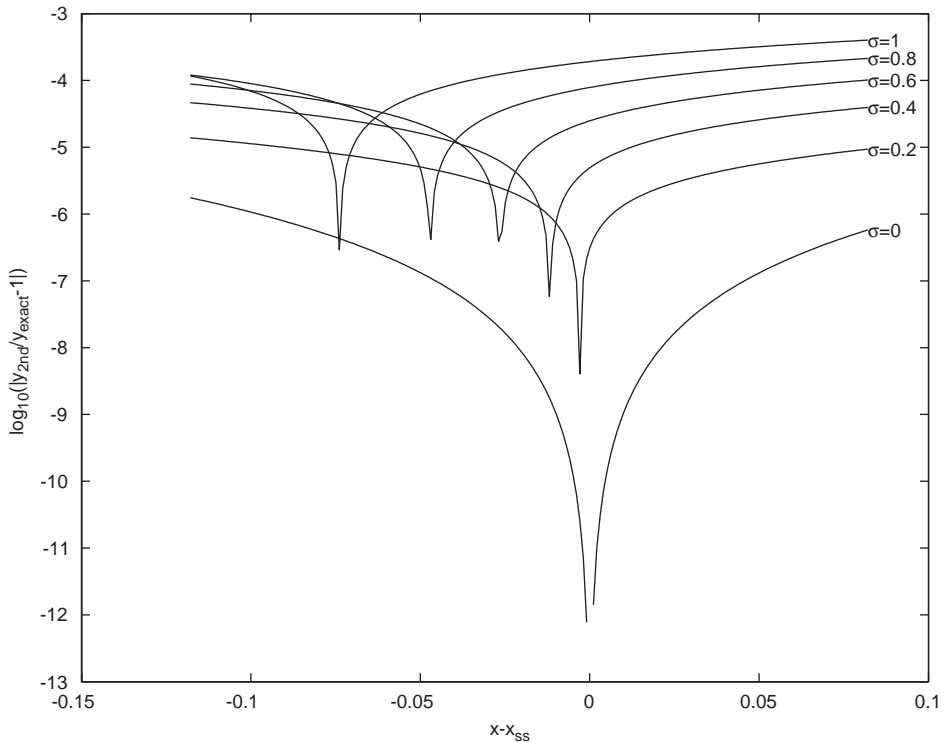


Fig. 1. Non-local accuracy test. Note:  $y$  denotes the price-dividend ratio,  $x$  (the state variable) denotes the log of the gross growth rate of dividends,  $\sigma$  denotes the parameter scaling the standard deviation of the exogenous shock, and  $x_{ss}$  denotes the steady-state value of  $x$ . All parameters take the values used in the benchmark calibration, as given in the note to Table 1. In computing the exact solution we truncated the infinite sum at  $i = 1000$ .

three different calibrations of the above asset-pricing model. To facilitate comparison, we reproduce in our Table 1 their numbers in the rows labeled fixed-point algorithm. We also report the correct coefficients, which can be obtained either by evaluating Eq. (10) or by running the programs implementing our proposed algorithm (asset\_run.m). The table shows that the discrepancy with respect to the Taylor expansion introduced by the fixed-point algorithm of Collard and Juillard can be significant. For example, when the intertemporal elasticity of substitution is low ( $\theta = -10$ ) the coefficients associated with the constant, linear, and quadratic terms are, respectively, 4, 24, and 24 percent larger than those of the second-order expansion. Similarly, when the underlying shock is assumed to be highly persistent ( $\rho = 0.9$ ), then the difference between the coefficients on the constant term is 34 percent and on the linear and quadratic terms is 16 percent.

We close this section by performing an accuracy test of the second-order approximation method described in this paper. The test is along the lines of Judd (1998). Specifically, let  $y^{2nd}(x, \sigma)$  denote the second-order approximation to the price-dividend

ratio given in Eq. (10) for some value of the state variable  $x$  and of the scale parameter  $\sigma$ . Similarly, let  $y^{\text{exact}}(x, \sigma)$  be the exact equilibrium price-dividend ratio given in Eq. (9). Fig. 1 plots  $\log_{10}(|y^{\text{2nd}}(x, \sigma)/y^{\text{exact}}(x, \sigma) - 1|)$  as a function of  $x - \bar{x}$  and  $\sigma$ . We explore a wide range of values for  $x$  and  $\sigma$ . In our calibration,  $\bar{x}$  takes the value 0.0179. We construct the figure using values for  $x$  between  $-0.1$  and  $0.1$ . The range for the scale parameter  $\sigma$  is  $[0, 1]$ . The maximum approximation error is of the order  $10^{-3.4}$ .

## 6. Conclusion

Most models used in modern macroeconomics are too complex to allow for exact solutions. For this reason, researchers have appealed to numerical approximation techniques. One popular and widely used approximation technique is a first-order perturbation method delivering a linear approximation to the policy function. One reason for the popularity of first-order perturbation techniques is that they do not suffer from the ‘curse of dimensionality.’ That is, problems with a large number of state variables can be handled without much computational demands. However, an important limitation of this approximation technique is that the solution displays the certainty equivalence property. In particular, the first-order approximation to the unconditional means of endogenous variables coincides with their non-stochastic steady state values. This limitation restricts the range of questions that can be addressed in a meaningful way using first-order perturbation techniques. Two such questions are welfare evaluations and risk premia in stochastic environments.

Within the family of perturbation methods an obvious way to overcome these limitations is to perform a higher-order approximation to the policy function. This is precisely what this paper accomplishes. We build on previous work by Collard and Juillard, Sims, and Judd among others. In particular, this paper derives a second-order approximation to the solution of a general class of discrete-time rational expectations models.

The main theoretical contribution of the paper is to show that for any model belonging to the general class considered, the coefficients on the terms linear and quadratic in the state vector in a second-order expansion of the decision rule are independent of the volatility of the exogenous shocks. In other words, these coefficients must be the same in the stochastic and the deterministic versions of the model. Thus, up to second order, the presence of uncertainty affects only the constant term of the decision rules. But the fact that only the constant term is affected by the presence of uncertainty is by no means inconsequential. For it implies that up to second order the unconditional mean of endogenous variables can in general be significantly different from their non-stochastic steady state values. Thus, second-order approximation methods can in principle capture important effects of uncertainty on average rate of return differentials across assets with different risk characteristics and on the average level of consumer welfare. An additional advantage of higher-order perturbation methods is that like their first-order counterparts, they do not suffer from the curse of dimensionality. This is because given the first-order approximation to the policy function, finding the

coefficients of a second-order approximation simply entails solving a system of linear equations.

In addition to fully characterizing the second-order approximation to the policy function analytically, the paper presents a set of MATLAB programs designed to compute the coefficients of the second-order approximation. This code is publicly available. The validity and applicability of the proposed method is illustrated by solving the dynamics of a number of model economies. Our computer code coexists with others that have been developed recently by Chris Sims and Fabrice Collard and Michel Juillard to accomplish the same task. We believe that the availability of this set of independently developed codes, which have been shown to deliver identical results for a number of example economies, helps build confidence across potential users.

A number of important aspects of higher-order approximations to the policy function remain to be explored. At the forefront stands the problem of characterizing second-order accurate approximations to artificial time series as well as to conditional and unconditional moments of endogenous variables. We leave this task for future research.

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