ANALYSIS OF THE ORDER FLOW AUCTION UNDER PROPOSER-BUILDER SEPARATION

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Abstract. In this paper, we consider the impact of the order flow auction (OFA) in the context of the proposer-builder separation (PBS) mechanism through a game-theoretic perspective. The OFA is designed to improve user welfare by redistributing maximal extractable value (MEV) to the users, in which two auctions take place: the order flow auction and the block-building auction. We formulate the OFA as a multiplayer game, and focus our analyses on the case of two competing players (builders). We prove the existence and uniqueness of a Nash equilibrium for the two-player game, and derive a closed-form solution by solving a quartic equation. Our result shows that the builder with a competitive advantage pays a relatively lower cost, leading to centralization in the builder space. In contrast, the proposer's shares evolve as a martingale process, which implies decentralization in the proposer (or, validator) space. Our analyses rely on various tools from stochastic processes, convex optimization, and polynomial equations. We also conduct numerical studies to corroborate our findings, and explore other features of the OFA under the PBS mechanism.

1. Introduction

A blockchain is a decentralized, distributed, and tamper-proof digital ledger that tracks and verifies digital transactions securely without the need for a central authority. Its applications span a wide range of industries, including sustainable energies [27], cryptocurrency [13, 15], healthcare [4, 17, 19], and construction industry [14].

To maintain its decentralized structure, blockchain relies on consensus mechanisms, with the two most widely used being Proof-of-Work (PoW) and Proof-of-Stake (PoS). PoW requires substantial computational power to solve cryptographic puzzles, making it highly energy-intensive. In contrast, PoS is more energy-efficient as it selects validators based on their stake rather than computational effort. Due to its energy efficiency and increasing adoption, we focus on the PoS mechanism in this study.

However, decentralization, the core principle of blockchain technology, faces challenges as large validators gain power in the PoS system. (Validators are often also called proposers; here we use both terms interchangeably.) In PoS, validators with larger holdings have a higher chance of validating new blocks and earning more rewards, increasing wealth concentration and excluding smaller participants [3].

Maximal Extractable Value (MEV). A significant portion of validators' revenue comes from MEV, which involves strategically reordering, inserting, or censoring transactions within a block to maximize profits. This includes activities like DEX arbitrage, sandwich attacks, and liquidations that exploit inefficiencies in the market. Some of these activities depend

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solely on the blockchain's state, using on-chain data to extract value. Others require information from external sources, such as off-chain data, to identify opportunities like CEX-DEX arbitrage [9, 25].

MEV is widely considered as one of the greatest threats to decentralization in blockchain networks, favoring the validators with more resources [7]. Extracting MEV effectively requires significant capital, advanced strategies, and considerable computational power, which most ordinary validators may not have access to. As a result, well-equipped validators can gain an advantage, further centralizing their control within the network [2, 6, 10, 12, 16, 28].

Proposer-Builder Separation (PBS). To distribute MEV fairly among the validators and to prevent centralization, Ethereum introduced the PBS mechanism [10]. Originally, validators were responsible for both proposing new blocks and constructing their contents. PBS separates these responsibilities into two distinct roles: **block builders** and **block proposers**.

Block proposers (or validators) validate and propose blocks; block builders become responsible for assembling blocks. Block builders compete to create the most profitable block, and participate in a block-building auction, offering fees (bids) to the proposer. The proposer then receives the block bundle created by the winning builder and their bid. This allows the proposers to collect significant auction revenue without requiring advanced technical expertise. By fostering competition among the builders, this system diminishes the advantage previously held by sophisticated validators over ordinary ones, facilitating a more balanced distribution of MEV among validators [5, 8, 10, 28].

While PBS can mitigate centralization among the validators to some extent, it tends to create centralization within the builder community. For instance, some builders are exceptionally skilled at exploiting arbitrage opportunities, enabling them to capture significant MEV and consistently win block production opportunities. This concentration of expertise and resources among a few builders can lead to a situation where only a small number of entities control a large portion of the block-building market, thus centralizing power within the builder space [2, 12].

Order Flow Auction (OFA). OFA is another mechanism aiming to mitigate centralization and redistribute MEV. While PBS involves the builders and the validators, OFA focuses on the interaction between the **users** and the **builders**. It seeks to return a portion of MEV back to the users. In this process, users send their orders to a third-party auction, where block builders or MEV-extracting searchers bid for the exclusive rights to execute strategies on these orders. This approach is conceptually analogous to Payment For Order Flow (PFOF) in traditional finance. Gosselin and Chiplunkar [11], and Ventures [25] outline the structure of OFA as follows:

(1) Order Flow Originators (OFO): Order Flow Originators (OFOs) refer to wallets, decentralized applications (dApps), or custodians that users interact with for on-chain transactions. These OFOs gather the orders created by users and forward them to the OFA.
(2) Auctioneer: OFA discloses certain information to a group of bidders.
(3) Bidders: Bidders then need to submit their bids back to the OFA. The OFA must determine the criteria for selecting the winning bids.

(4) Winning Bid: Finally, the OFA bundles are sent to the winning bidder for inclusion in the blockchain.

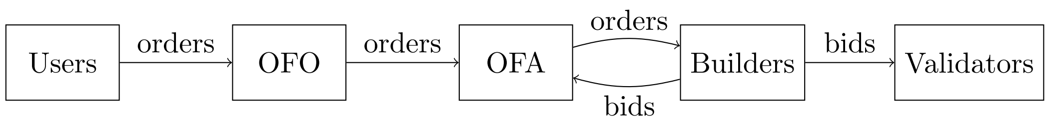


Figure 1. OFA

Figure 1 presents a flowchart illustrating the OFA process. It is important to note that an order included by a block builder is not necessarily guaranteed on-chain inclusion. The order will only be included if the block builder wins the block-building auction. Therefore, the OFA must provide reliable inclusion guarantees and faces the challenge of the double auction problem, which stems from the interaction between the OFA and the block-building auction [11]. If higher rebates are offered to the users, less MEV can be redistributed to the validators, which may affect the timely inclusion of blocks. Thus, it is crucial to find the optimal balance between user rebates and inclusion fees offered to the validators in the winner selection process [11].

Main Contributions.

(1) We formulate the OFA as an M-player game, where each builder's decision variable is the amount of MEV they are willing to pay. Under suitable conditions, we establish that the Nash equilibrium can be characterized by the first-order conditions.
(2) We study in depth the case when $M = 2$, i.e., there are two competing builders. Interestingly, solving for the corresponding Nash equilibrium boils down to solving a univariate quartic equation. We show that there exists a unique Nash equilibrium and derive its closed-form solution. Our analysis reveals that when there are two builders, their equilibrium payments do not scale linearly with the MEV they can extract. Instead, the more capable builder pays relatively less, allowing them to earn higher expected revenue, which drives centralization. Our simulation experiments with three players show a similar pattern, with an interesting variation: the most capable player's advantage is further amplified, while the gap between the second and the least capable players narrows compared to the two-player case.
(3) We formulate the evolution of validators' stake shares as a Pólya urn process with a random replacement matrix. Under this framework, we establish that the stake shares follow a martingale process, aligning with our simulation results, which show that the average stake shares remain nearly constant over time. Furthermore, we analyze the long-term behavior of this process. In the absence of consumption factors (i.e., when no costs are incurred due to staking), we characterize the distribution of the limiting stake shares via functional equations.

Literature Review. Our research contributes to the literature on blockchain centralization. Prior studies have explored various factors driving centralization in blockchain ecosystems, both with and without the PBS mechanism. Reward heterogeneity, a direct cause of centralization, arises from skill disparities among block producers in the absence of PBS [2]. It is further influenced by order flow acquisition, along with the resulting MEV extraction and arbitrage opportunities under PBS [6, 12]. In fact, the role of order flow in centralization has been widely recognized. Empirical studies show that private order flow exacerbates disparities in block-building capacity among builders [28], and those with greater access to private order flow have a higher probability of winning the block-buildling auction [26].

Our research takes into account both the builder heterogeneity in skills and knowledge, and the process of order flow acquisition. Relative to previous studies, we further model order flow acquisition as an auction that explicitly captures its relationship with builders' block-building capacity, while also incorporating the interaction between the order flow auction and the block-building auction, which remains under-explored in previous studies.

Our research also contributes to the literature on the evolution of validators' stake shares in the Proof-of-Stake (PoS) system. Rosu and Saleh [18], and Tang [20] show that in a standard PoS system, validators' stake shares evolve as a martingale process. Tang and Yao [22] analyze stake share evolution under a polynomial voting rule. See also Tang [21] for a review. Building on these works, our research further examines a setting in which the validator's reward for proposing a block is stochastic and dependent on builders' bids. In addition, we include consumption factors into the system, recognizing that staking in the pool may incur costs, including opportunity costs, locked funds, and other potential expenses.

Organization of the paper: The remainder of the paper is organized as follows. In Section 2, we present the models of the OFA and PBS. In Section 3, we analyze the Nash equilibrium among the builders, and in Section 4, we study the evolution of stake shares of the validators. We present numerical findings in Section 5. Finally, we conclude with Section 6.

2. The OFA and PBS Models

In this section, we develop a formal model for the OFA, PBS, and the PoS system, focusing on the equilibrium and stochastic processes associated with the model. Section 2.1 introduces the model for builders, who participate in both the order flow auction and the block-building auction. Section 2.2 presents the model for validators and the PoS system.

First, here is a list of some of the common notations used throughout the paper.

- $\mathbb{N}_+$ denotes the set of positive integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}_+$ denotes the set of positive real numbers.
- $[n]$ denotes the set $\{1, 2, \ldots, n\}$.
- $a = \mathcal{O}(b)$ means $\frac{a}{b}$ is bounded from above as $b \to \infty$; $a = \Theta(b)$ means $\frac{a}{b}$ is bounded from below and above as $b \to \infty$; and $a = o(b)$ or $b \gg a$ means $\frac{a}{b}$ decays towards zero as $b \to \infty$.
- $\mathbf{I}(A)$ denotes the indicator function of event A, which equals 1 if event A happens and 0 otherwise.

Builders in a decentralized system are responsible for assembling transaction blocks. They participate in two auctions: the order flow auction and the block-building auction. In the order flow auction, builders compete to acquire users' order flow, which can provide additional MEV opportunities. By strategically integrating the auctioned order with their existing order flow, they can optimize execution for increased profitability. In the block-building auction, builders bid for the right to propose their assembled block for inclusion on-chain, thereby capturing MEV generated from transaction sequencing and execution.

Validators are responsible for validating and proposing blocks. They participate in the consensus process by staking cryptocurrencies as collateral. At each block-building opportunity, a validator is selected to propose the next block. The selected validator receives the bid from the winning builder in the block-building auction.

Remark 2.1. In our model, we do not distinguish between MEV searchers and builders, but instead treat them as a collective entity. This is motivated by the fact that searchers often direct their order flow to dominant builders or restrict it exclusively to vertically integrated builder-searcher entities [12, 23, 24, 28]. This close coordination, coupled with their shared objective of maximizing MEV, has convinced us to ignore any distinction between their roles, and to focus instead on their combined strategic behavior in the context of MEV extraction.

Let $M \in \mathbb{N}_+$ be the total number of builders, and $N \in \mathbb{N}_+$ be the total number of validators, which will stay fixed throughout the paper; and let $[M]$ and $[N]$ denote the sets of all builders and all validators, respectively. Time is modeled as a discrete sequence indexed by $t = 0, 1, 2, \ldots$. In each period t, a single round of the order flow auction and a single round of the block-building auction take place.

2.1. **Builder's Game.** Let $f_{i,t}$ represent the amount of Maximal Extractable Value (MEV) that builder i can capture at round t. Assume that $f_{i,t}$ follows a distribution $D_{i,t}$ for each $i \in [M]$ and for each $t = 0, 1, 2, \ldots$. The expected value of $f_{i,t}$ is given by $\mathbf{E}[f_{i,t}] = \bar{f}_{i,t}$. Suppose builder i has value $v_{i,t}$ for winning the transaction right in the order flow auction at round t. Assume that $v_{i,t}$ follows some distribution $F_{i,t}$, for $i \in [M]$ and $t = 0, 1, 2, \ldots$. The expected value of $v_{i,t}$ is given by $\mathbf{E}[v_{i,t}] = \bar{v}_{i,t}$.

Let $h_{i,t}$ denote the total amount that builder i is willing to pay to both users and validators at round t. Suppose a fraction $\mu h_{i,t}$ is ultimately paid to users, and $(1-\mu)h_{i,t}$ is paid to the selected validator, with μ determined by the auction design ($0 \le \mu \le 1$). We make the following assumptions.

Assumption 2.2. *Builders' abilities to extract MEV are time-homogeneous.*

Assumption 2.3. *The winners of the order flow auction and the block-building auction are determined independently.*

Remark 2.4. We view builders' capacities to optimize MEV extraction as intrinsic characteristics that remain relatively stable over time. This perspective is in a similar spirit to that of Bahrani et al. [2]. We will omit the time index in the following discussions of builders.

In each auction, the winner is selected randomly, with builder i winning the order flow auction with probability $\frac{h_i}{\sum_{j=1}^{M} h_j}$ and winning the block-building auction with the same probability. Let V_i denote the event that builder i wins the order flow auction, and Z_i denote the event that builder i wins the block-building auction:

$$\mathbf{I}(V_i) = \begin{cases} 1 & \text{if builder } i \text{ wins the order flow auction,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{I}(Z_i) = \begin{cases} 1 & \text{if builder } i \text{ wins the block-building auction,} \\ 0 & \text{otherwise.} \end{cases}$$

Table 1 presents the four possible outcomes of the two auctions for builder i, along with the corresponding revenue (utility) in each case. (Revenue can be regarded as utility, and we use both terms interchangeably in this paper.) Let h_{-i} denote the $M-1$ strategies of all the builders except i. The expected utility of builder i is given by:

$$\pi_i(h_i|h_{-i}) = \mathbf{E}\left[\mathbf{I}\left(Z_i\right)\left(f_i + v_i\mathbf{I}\left(V_i\right)\right) - \mu h_i \mathbf{I}\left(V_i\right) - (1-\mu)h_i\mathbf{I}\left(Z_i\right)\right],$$
$$= \bar{f}_i \frac{h_i}{\sum_{j=1}^M h_j} + \bar{v}_i\left(\frac{h_i}{\sum_{j=1}^M h_j}\right)^2 - \frac{h_i^2}{\sum_{j=1}^M h_j}. \tag{2.1}$$

Each builder i chooses a bid h_i from the strategy space $B_i := (0,\infty)$ to maximize their expected utility. Let h_i^* denote the optimal bid, i.e.,

$$h_i^* = \arg\max_{h_i>0} \bar{f}_i \frac{h_i}{\sum_{j=1}^M h_j} + \bar{v}_i\left(\frac{h_i}{\sum_{j=1}^M h_j}\right)^2 - \frac{h_i^2}{\sum_{j=1}^M h_j}.$$

In contrast to Capponi et al. [6], which introduce an additional player, the order flow provider, and model the order flow acquisition process using a quadratic function, our framework takes a different perspective. Specifically, we model the payment as a proportion of the MEV obtained and introduce randomness in the winner selection process for both the OFA and PBS. As a result, our objective function involves the term $\frac{h_i}{\sum_{j=1}^M h_j}$, reflecting the competitive interaction among the builders.

Order Flow Auction	Block-Building Auction	Revenue
Win	Win	$f_i + v_i - h_i$
Win	Lose	$-\mu h_i$
Lose	Win	$f_i - (1-\mu)\, h_i$
Lose	Lose	0

Table 1. Revenue Outcomes for Builder i.

We aim to find the Nash equilibrium of the game among the builders.

Definition 2.5. *Let M denote the total number of builders. Let B_i be the set of all possible strategies for builder i, where $i \in [M]$. Let $h = (h_i, h_{-i})$ be a strategy profile where h_{-i} denotes the $M-1$ strategies of all the builders except i. A Nash equilibrium is a strategy profile $h^* = (h_i^*, h_{-i}^*)$ if*

$$\pi_i(h_i^*|h_{-i}^*) \geq \pi_i(h_i|h_{-i}^*)$$

for all $h_i \in B_i$.

2.2. **Validator's Game.** Let $s_{j,t}$ denote the stake held by validator j at time t, and define the total stake at time t as $S_t := \sum_{j=1}^n s_{j,t}$. The fraction of the total stake held by validator j at time t is given by $\omega_{j,t} := \frac{s_{j,t}}{S_t}$ for $j \in [N]$. In each round t, the probability that validator j is selected to propose a block is $\omega_{j,t-1}$. The initial stake share of validator j is given by $\omega_{j,0} = \frac{s_{j,0}}{S_0}$, where $s_{j,0}$ represents the initial stake held by validator j, and $S_0 = \sum_{j=1}^N s_{j,0}$ denotes the total initial stake held by all N validators. We assume that each validator holds a positive initial stake.

Assumption 2.6. *For all $j \in [N]$, the initial stake is strictly positive, i.e., $s_{j,0} > 0$, and hence the total initial stake satisfies $S_0 = \sum_{j=1}^{N} s_{j,0} > 0$.*

If selected, a validator can either propose the block submitted by the winning builder in the block-building auction or choose to propose the block built by themselves. Let $\beta_{w,t}$ denote the bid submitted by the winning builder in the block-building auction at time t. Since we assume that builders' MEV extraction capacities are time-homogeneous, we omit the subscript t and refer to it as β_w in the rest of the paper. The value of β_w is drawn from the set $\{\mu h_1, \mu h_2, \ldots, \mu h_M\}$, where it takes the value μh_i with probability $\frac{h_i}{\sum_{k=1}^{M} h_k}$. Let $\beta_{v,t}$ denote the value of the block built directly by the selected validator at time t. We make the following assumptions:

Assumption 2.7. *Validators' MEV extraction abilities are time-homogeneous.*

Assumption 2.8. *MEV extraction abilities are identical across all validators and are characterized by the same value β_v.*

Remark 2.9. Similar to the builders, we consider a validator's ability to extract MEV as an intrinsic characteristic that remains relatively stable over time. In the remainder of the paper, we omit the time subscript t in $\beta_{v,t}$ and refer to it simply as β_v. Moreover, the validators with significantly higher skills in MEV extraction would likely operate as the builders instead. Consequently, those in the role of the validators are expected to exhibit relatively homogeneous MEV extraction capabilities in the PBS system.

The reward received by the selected validator for proposing a block is given by $R_t = \max\{\beta_w, \beta_v\}$. $\{R_t\}_{t\geq 1}$ is a sequence of i.i.d random variables with mean $\mathbb{E}(R_t) = R$ for all $t \geq 1$.

Additionally, a staking cost $\alpha \frac{s_{j,t}}{S_t^{1+\gamma}}$ is incurred, where $\gamma \geq 0$. This cost may reflect factors such as opportunity cost of locked funds or operational expenses. It is proportional to a validator's stake share, conceptually similar to transaction fees in traditional finance, which are typically a small percentage of the transaction amount.

Let $R_{\min} \coloneqq \min\{k_1, k_2, \ldots, k_M\}$ denote the minimum value of the i.i.d variable R_t, where $k_i = \max\{\beta_v, \mu h_i\}$ for $i \in [M]$. To ensure that α is sufficiently small, we make the following assumption.

Assumption 2.10. $\alpha < \min\{S_0^{\gamma} R_{\min},\, S_0,\, R_{\min} + c\}$*, where $c > 0$ is a constant.*

This assumption guarantees that S_t is strictly increasing for all $t \geq 0$ (see Lemma 4.2).

Let $X_{j,t}$ denote the event that validator j is chosen at time t, and define its corresponding indicator variable $\mathbf{1}(X_{j,t})$ as

$$\mathbf{1}(X_{j,t}) = \begin{cases} 1, & \text{if validator } j \text{ is selected at time } t, \\ 0, & \text{otherwise.} \end{cases}$$

Assumption 2.11. *The event $X_{j,t}$, i.e., the selection of validator j at time t, is independent of all other random variables in the system.*

The stake held by each validator evolves according to the following update rule:

$$s_{j,t} = s_{j,t-1} + R_t \mathbf{1}\left(X_{j,t}\right) - \alpha \frac{s_{j,t-1}}{S_{t-1}^{1+\gamma}},$$
$$= \begin{cases} s_{j,t-1} - \alpha \frac{s_{j,t-1}}{S_{t-1}^{1+\gamma}} & \text{with probability } 1-\omega_{j,t-1} \\ s_{j,t-1} + R_t - \alpha \frac{s_{j,t-1}}{S_{t-1}^{1+\gamma}} & \text{with probability } \omega_{j,t-1} \end{cases} \quad \text{for } j \in [N]. \tag{2.2}$$

As a result, the total stake S_t evolves as

$$S_t = S_{t-1} + R_t - \frac{\alpha}{S_{t-1}^{\gamma}}, \tag{2.3}$$

where we take into account the identity $\sum_{k=1}^{n} s_{k,t} = S_t$. Finally, for each $t \in \mathbb{N}_+$, let $\mathcal{F}_t$ denote the filtration generated by random events $(X_{j,r} : j \in [N], r \le t)$.

3. Analysis of the game of the builders

In this section, we analyze the strategic interactions among the builders. Section 3.1 considers the general setting with M players, and the two-player game is studied in Section 3.2. Numerical results for the multi-player setting are given in Section 5.1.

3.1. Nash Equilibrium among M Builders.

We make the following assumptions about the game between builders.

Assumption 3.1. *The parameters satisfy $\bar{f}_i > 0$, $\bar{v}_i > 0$, and $\bar{f}_i \ge \bar{v}_i$ for all $i \in [M]$*

Remark 3.2. This assumption implies that, in expectation, the MEV each builder can independently extract is at least as large as the additional MEV that may be obtained from the order being auctioned in the OFA. In other words, the orders auctioned in the OFA should not constitute the primary source of MEV in expectation.

Lemma 3.3. *The payoff function $\pi_i(h_i|h_{-i})$ is concave with respect to builder i's own strategy h_i.*

Proof. Let $H = \sum_j h_j$ and $H_{-i} = \sum_{j \ne i} h_j$. It suffices to note that

$$\frac{\partial^2 \pi_i}{\partial h_i^2} = -\frac{2H_{-i}\left(\bar{f}_i H + H_{-i}\left(H_{-i} - \bar{v}_i\right) + h_i\left(H_{-i} + 2\bar{v}_i\right)\right)}{H^4} \le 0,$$

holds for all $h_i \in B_i$ when $\bar{f}_i \ge \bar{v}_i$ (Assumption 3.1). □

Since the utility function is concave and continuous, the Nash equilibrium of the game can be characterized by the first-order conditions.

Lemma 3.4. *The first-order conditions for the Nash equilibrium of the game $\langle \mathcal{I}, (B_i)_{i\in\mathcal{I}}, (\pi_i)_{i\in\mathcal{I}} \rangle$, where $\mathcal{I} = [M]$, are given by:*

$$\frac{\partial \pi_i}{\partial h_i} = 0 \quad \text{for } i \in [M],$$

where

$$\frac{\partial \pi_i}{\partial h_i} = \frac{-\bar{f}_i h_i H + \bar{f}_i H^2 + 2\bar{v}_i h_i H - 2h_i^2 \bar{v}_i - 2h_i H^2 + h_i^2 H}{H^3} \quad \text{for } i \in [M], \tag{3.1}$$

with $H = \sum_j h_j$.

In the remaining of this section, we focus on the case where $M = 2$.

3.2. $M = 2$ **Builders.** Given that $h_1, h_2 \neq 0$, let $h_2 = \lambda h_1$, where $\lambda > 0$. Substituting this into Eqs. (3.4)–(3.1), the equations simplify to:

$$\begin{cases} h_1^3 (1+2\lambda)(1+\lambda) = \lambda h_1^2 \left(\bar{f}_1 \lambda + \bar{f}_1 + 2\bar{v}_1\right), \\ h_1^3 \lambda (2+\lambda)(1+\lambda) = h_1^2 \left(\bar{f}_2 + \left(\bar{f}_2 + 2\bar{v}_2\right)\lambda\right). \end{cases} \tag{3.2}$$

Solving Eq. (3.2) is equivalent to solving the quartic equation:

$$P(\lambda) = \bar{f}_1 \lambda^4 + \left(3\bar{f}_1 + 2\bar{v}_1\right)\lambda^3 + \left(2\bar{f}_1 - 2\bar{f}_2 + 4\bar{v}_1 - 4\bar{v}_2\right)\lambda^2 - \left(3\bar{f}_2 + 2\bar{v}_2\right)\lambda - \bar{f}_2 = 0. \tag{3.3}$$

Once λ is obtained, we set $h_2 = \lambda h_1$ and solve for h_1 using either of the reformulated equations:

$$h_1 = \frac{\lambda\left(\bar{f}_1 \lambda + \bar{f}_1 + 2\bar{v}_1\right)}{(1+2\lambda)(1+\lambda)} = \frac{\bar{f}_2 + \left(\bar{f}_2 + 2\bar{v}_2\right)\lambda}{\lambda(2+\lambda)(1+\lambda)}. \tag{3.4}$$

Theorem 3.5. *The game $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (\pi_i)_{i\in\mathcal{I}}\rangle$, where $\mathcal{I} = \{1, 2\}$, has a unique Nash equilibrium $h^* = (h_1^*, h_2^*)$.*

Proof. We analyze the solutions to Eq. (3.3). By Vieta's formulas, the products of the roots satisfies $\lambda_1\lambda_2\lambda_3\lambda_4 = -\frac{\bar{f}_2}{\bar{f}_1} < 0$, indicating the roots of Eq. (3.3) must satisfy one of the following: (1) one positive real root and three negative real roots, (2) three positive real roots and one negative real root, or (3) one positive real root, one negative real root, and two complex real roots. We will eliminate cases (2) and (3), thereby proving the existence of a unique positive real root.

Assume, for contradiction, that the roots consist of three positive positive real roots, denoted $\lambda_1, \lambda_2, \lambda_3$, and one negative real root, denoted λ_4. We further observe from Vieta's formulas that

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -3 - \frac{2\bar{v}_1}{\bar{f}_1} < -3, \\ \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} = -3 - \frac{2\bar{v}_2}{\bar{f}_2} < -3. \end{cases}$$

Since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 0$, it follows that $\lambda_3 + \lambda_4 < 0$. However, this implies $\frac{1}{\lambda_3} + \frac{1}{\lambda_4} > 0$, contradicting $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} < 0$. This contradiction eliminates case (2).

To eliminate case (3), we observe that $P(0) = -\bar{f}_2 < 0$, $P(-\frac{1}{2}) = \frac{3}{16}\bar{f}_1 + \frac{3}{4}\bar{v}_1 > 0$, and $P(-2) = -3\bar{f}_2 - 12\bar{v}_2 < 0$. Since $P(\lambda)$ is continuous in λ, there exists one root in the interval $\left(-\frac{1}{2}, 0\right)$ and another root in the interval $\left(-2, -\frac{1}{2}\right)$ by the intermediate value theorem. Thus, the polynomial has at least two negative real roots, leaving no possibility for two complex roots.

Having ruled out cases (2) and (3), we conclude that the only remaining possibility is case (1), where there exists exactly one positive real root, and it is unique.

We now prove that the unique positive solution to Eq. (3.3), denoted by λ^*, corresponds to the unique Nash equilibrium of the game. Let $h_1^* = \dfrac{\lambda^*\left(\bar{f}_1\lambda^* + \bar{f}_1 + 2\bar{v}_1\right)}{(1+2\lambda^*)(1+\lambda^*)}$ and $h_2^* = \lambda^* h_1^*$. By Lemma 3.3, the strategy profile (h_1^*, h_2^*) is a Nash equilibrium. We now establish its uniqueness. Suppose, for the sake of contradiction, that there exists another Nash equilibrium $h' = (h_1', h_2') \neq (h_1^*, h_2^*)$ such that $\pi_i(h_i'|h_j') \geq \pi_i(h_i|h_j')$ for all $h_i \in S_i$ and for $i, j \in \{1, 2\}, i \neq$

j. By Lemma 3.3, h_1' must satisfy $\frac{\partial \pi_1}{\partial h_1}\Big|_{h_2=h_2'} = 0$, and h_2' must satisfy $\frac{\partial \pi_2}{\partial h_2}\Big|_{h_1=h_1'} = 0$. Consequently, $h' = (h_1', h_2')$ would also satisfy Eq. (3.2). However, from the earlier analysis, we proved that Eq. (3.2) admits a unique solution, namely (h_1^*, h_2^*). Therefore, it must hold that $h_1' = h_1^*$ and $h_2' = h_2^*$. Thus, $h^* = (h_1^*, h_2^*)$ is the unique Nash equilibrium of the game. □

After establishing the existence and uniqueness of the Nash equilibrium, we derive its closed-form solution. The proof relies on a computer-assisted technique.

Theorem 3.6. *The unique Nash equilibrium is given by:*

$$\begin{aligned} \lambda^* &= -\frac{3\bar{f}_1 + 2\bar{v}_1}{4\bar{f}_1} + S + \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}, \\ h_1^* &= \frac{\lambda^*(\bar{f}_1\lambda^* + \bar{f}_1 + 2\bar{v}_1)}{(1+2\lambda^*)(1+\lambda^*)} = \frac{\bar{f}_2 + (\bar{f}_2 + 2\bar{v}_2)\lambda^*}{\lambda^*(2+\lambda^*)(1+\lambda^*)}, \\ h_2^* &= \lambda^* h_1^*, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} p &= -\frac{11\bar{f}_1^{\,2} + 12\bar{v}_1^{\,2} + 4\bar{f}_1(4\bar{f}_2 + \bar{v}_1 + 8\bar{v}_2)}{8\bar{f}_1^{\,2}}, \\ q &= \frac{3\bar{f}_1^{\,3} + 8\bar{v}_1^{\,3} + 4\bar{f}_1\bar{v}_1(4\bar{f}_2 + \bar{v}_1 + 8\bar{v}_2) + \bar{f}_1^{\,2}(-10\bar{v}_1 + 32\bar{v}_2)}{8\bar{f}_1^{\,3}}, \\ \Delta_0 &= (2\bar{f}_1 - 2\bar{f}_2 + 4\bar{v}_1 - 4\bar{v}_2)^2 + 3(3\bar{f}_1 + 2\bar{v}_1)(3\bar{f}_2 + 2\bar{v}_2) - 12\bar{f}_1\bar{f}_2, \\ \Delta_1 &= -27\bar{f}_2(3\bar{f}_1 + 2\bar{v}_1)^2 + 72\bar{f}_1\bar{f}_2(2\bar{f}_1 - 2\bar{f}_2 + 4\bar{v}_1 - 4\bar{v}_2) + 2(2\bar{f}_1 - 2\bar{f}_2 + 4\bar{v}_1 - 4\bar{v}_2)^3 \\ &\quad + 9(3\bar{f}_1 + 2\bar{v}_1)(2\bar{f}_1 - 2\bar{f}_2 + 4\bar{v}_1 - 4\bar{v}_2)(3\bar{f}_2 + 2\bar{v}_2) + 27\bar{f}_1(3\bar{f}_2 + 2\bar{v}_2)^2, \\ \varphi &= \arccos\left(\frac{\Delta_1}{2\sqrt{\Delta_0^3}}\right), \\ S &= \frac{1}{2}\sqrt{-\frac{2}{3}p + \frac{2}{3\bar{f}_1}\sqrt{\Delta_0}\cos\frac{\varphi}{3}}. \end{aligned} \tag{3.6}$$

Proof. Since all the roots of Eq. (3.3) are real and exactly one of them is positive, the four real roots can be expressed as follows:

$$\begin{aligned} \lambda_{1,2} &= -\frac{3\bar{f}_1 + 2\bar{v}_1}{4\bar{f}_1} - S \pm \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}}, \\ \lambda_{3,4} &= -\frac{3\bar{f}_1 + 2\bar{v}_1}{4\bar{f}_1} + S \pm \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}. \end{aligned}$$

where p, q, S are defined in Eqs. (3.6). We want to show that $-S + \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}} < S + \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}$, which identifies the positive (and largest) real root of the quartic equation as Eq. (3.5). To prove this, observe that:

$$\begin{aligned} & -S + \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}} < S + \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}, \\ \iff & \sqrt{-4S^2 - 2p + \frac{q}{S}} < \ 4S + \sqrt{-4S^2 - 2p + \frac{q}{S}}, \\ \iff & 8S^3 - q + 4S^2\sqrt{-4S^2 - 2p - \frac{q}{S}} > 0. \end{aligned} \tag{3.7}$$

Here we rely on numerical computations. We find that the optimal value of the following minimization problem is approximately 3 (> 0):

$$\begin{aligned} \min_{\bar{f_1}, \bar{f_2}, \bar{v_1}, \bar{v_2} \in \mathbb{R}^+} \quad & 8S^3 - q, \\ \text{such that} \quad & \bar{f_1} \geq \bar{v_1}, \\ & \bar{f_2} \geq \bar{v_2}. \end{aligned} \tag{3.8}$$

(The implementation details and code are provided in Appendix A.) This ensures that the inequality (3.7) holds, and the theorem is proved. □

We then explore how the equilibrium solutions vary when we change the parameters $\bar{f_i}$ and $\bar{v_i}$ for $i = 1, 2$. Specifically, we examine whether, when one player is k times more capable than the other in extracting MEV, the ratio of their equilibrium payments also equals k.

Proposition 3.7. *Let k_1, k_2 be given such that $\frac{\bar{v_1}}{\bar{f_1}} = \frac{\bar{v_2}}{\bar{f_2}} = k_1$ and $\frac{\bar{f_1}}{\bar{f_2}} = k_2$, where $0 < k_1 \leq 1$ as given by Assumption 3.1. Then the following statements hold:*

- *If $k_2 < 1$, then $\frac{h_1^*}{h_2^*} > k_2$.*
- *If $k_2 = 1$, then $\frac{h_1^*}{h_2^*} = k_2 = 1$, and $h_1^* = h_2^* = \frac{\bar{f_1} + \bar{v_1}}{3} = \frac{\bar{f_2} + \bar{v_2}}{3}$.*
- *If $k_2 > 1$, then $\frac{h_1^*}{h_2^*} < k_2$.*

Proof. Since $\frac{h_1}{h_2} = \frac{1}{\lambda}$, proving the theorem is equivalent to proving that the largest real root of Eq. (3.3), denoted by λ_4, satisfies

$$\lambda_4 < \frac{1}{k_2} \text{ for } k_2 < 1, \ \lambda_4 = \frac{1}{k_2} = 1 \text{ for } k_2 = 1, \ \lambda_4 > \frac{1}{k_2} \text{ for } k_2 > 1.$$

Given that Eq. (3.3) satisfies $P(0) = -\bar{f_2} < 0$, it follows that proving the desired inequalities for λ_4 is equivalent to verifying that $P(\frac{1}{k_2})$ is positive for $k_2 < 1$, zero for $k_2 = 1$, and negative for $k_2 > 1$. Since $P(\lambda)$ is continuous in λ and $P(0) < 0$, it suffices to establish the sign of $P(\frac{1}{k_2})$.

Rewriting $P(\lambda)$ in terms of k_1, k_2, we obtain:

$$P(\lambda; k_1, k_2) = \bar{f_2}\left[k_2\lambda^4 + (3k_2 + 2k_1k_2)\lambda^3 + (2k_2 - 2 + 4k_1k_2 - 4k_1)\lambda^2 - (3 + 2k_1)\lambda - 1\right] = 0.$$

Define $G(k_1, k_2)$ as the evaluation of $P(\lambda; k_1, k_2)$ at $\lambda = \frac{1}{k_2}$.

$$G(k_1, k_2) = P(\lambda = \frac{1}{k_2}; k_1, k_2) = \frac{\bar{f_2}}{k_2^3}\left[1 + (2k_2 + 1 + 4k_1k_2 - 2k_1)k_2 - (3 + 2k_1)k_2^2 - k_2^3\right].$$

Differentiating with respect to k_1, we obtain

$$\frac{\partial G(k_1,k_2)}{\partial k_1} = \frac{2\bar{f}_2(k_2-1)}{k_2^2}.$$

Since $\frac{\partial G(k_1,k_2)}{\partial k_1} < 0$ for all $k_1 \in (0,1]$ when $k_2 \in (0,1)$, it follows that $G(k_1,k_2) > G(1,k_2)$ for all $k_1 \in (0,1)$. Evaluating $G(1,k_2)$, we find

$$G(1,k_2) = \frac{\bar{f}_2}{k_2^3}\left(1+k_2^2-k_2-k_2^3\right) = \frac{\bar{f}_2}{k_2^3}\left(1-k_2\right)\left(1+k_2^2\right),$$

which is positive for $k_2 \in (0,1)$. Therefore, $G(k_1,k_2) > 0$ for all $k_1 \in (0,1]$ and $k_2 \in (0,1)$.

Similarly, for $k_2 > 1$, we have $\frac{\partial G(k_1,k_2)}{\partial k_1} > 0$ for all $k_1 \in (0,1]$, implying that $G(k_1,k_2) < G(1,k_2)$ for all $k_1 \in (0,1)$. Since $G(1,k_2) = \frac{\bar{f}_2}{k_2^3}\left(1-k_2\right)\left(1+k_2^2\right) < 0$ for $k_2 > 1$, it follows that $G(k_1,k_2) < 0$ in this case.

For $k_2 = 1$, direct substitution yields $G(k_1, k_2 = 1) = 0$, which implies that $P(\lambda = 1; k_1, k_2) = 0$, and hence $\lambda = 1$ is a root of $P(\lambda)$. Consequently, it follows that $h_1 = h_2$. Additionally, when $k_2 = 1$, we have $\bar{f}_1 = \bar{f}_2$, $\bar{v}_1 = \bar{v}_2$. Substituting these identities into Eq. (3.4), we obtain:

$$h_1 = h_2 = \frac{\bar{f}_1+\bar{v}_1}{3} = \frac{\bar{f}_2+\bar{v}_2}{3}.$$

This completes the proof. □

Proposition 3.7 shows that when a player's ability to extract MEV, both with and without the order being auctioned in the order flow auction, is k times that of the other player (in expectation), their equilibrium payments do not scale proportionally with k. In particular, if player 2 can obtain more MEV, and this amount is k times that of player 1 in expectation, then player 2's equilibrium payment is less than k times the payment of player 1.

For example, consider the case where $\bar{f}_1 = 100, \bar{f}_2 = 200, \bar{v}_1 = 40$, and $\bar{v}_2 = 80$. At equilibrium, the corresponding bids are $h_1 = 49.94$ and $h_2 = 82.17$. After a single round of the game, player 1's expected revenue is 24.64, while player 2's expected revenue is 104.23. Although player 2's MEV extraction capability is only twice that of player 1, they ultimately earn more than four times player 1's revenue, in expectation. This illustrates that the gap between their MEV extraction abilities is amplified by the interaction between the order flow auction and the block-building auction.

For $M = 3$, we conduct numerical experiments to study the equilibrium solutions, as detailed in Section 5.1.

4. Evolution of Validators' Stake Shares

In this section, we analyze the dynamics of validators' stake shares over time. Each time a validator is selected to propose a block, their reward comes either from the bid submitted by the winning builder or from the MEV they extract by constructing the block themselves.

Theorem 4.1. *Validator j's stake share, $(\omega_{j,t})_{t\geq 0}$, is a martingale, and the limit $\omega_{j,\infty} := \lim_{t\to\infty}\omega_{j,t}$ exists almost surely, with $\mathbf{E}(\omega_{j,\infty}) = \omega_{j,0}$.*

Proof. For each j,

$$\begin{aligned}\omega_{j,t+1} &= \frac{s_{j,t+1}}{S_{t+1}},\\ &= \frac{s_{j,t} + R_{t+1}\mathbf{1}\left(X_{j,t+1}\right) - \alpha \frac{s_{j,t}}{S_t^{1+\gamma}}}{S_{t+1}},\\ &= \omega_{j,t}\frac{S_t - \alpha/S_t^{\gamma}}{S_{t+1}} + \frac{R_{t+1}}{S_{t+1}}\mathbf{1}\left(X_{j,t+1}\right).\end{aligned} \tag{4.1}$$

Noting that $\mathbf{E}\left[\mathbf{1}\left(X_{j,t+1}\right)|\mathcal{F}_t\right] = \omega_{j,t}$, we have

$$\begin{aligned}\mathbf{E}\left[\omega_{j,t+1}|\mathcal{F}_t\right] &= \omega_{j,t}\mathbf{E}\left[\frac{S_t - \alpha/S_t^{\gamma} + R_{t+1}}{S_{t+1}}\middle|\mathcal{F}_t\right],\\ &= \omega_{j,t}.\end{aligned} \tag{4.2}$$

where Eq. (4.2) holds because $S_{t+1} = S_t - \alpha/S_t^{\gamma} + R_{t+1}$. This shows that $\left(\omega_{j,t}\right)_{t\geq 0}$ is a martingale. Since it's a non-negative martingale, the martingale convergence theorem ensures that the limit $\omega_{j,\infty} := \lim_{t\to\infty}\omega_{j,t}$ exists almost surely. Furthermore, as $\omega_{j,t}$ is bounded, the bounded convergence theorem implies that $\mathbf{E}\left(\omega_{j,\infty}\right) = \lim_{t\to\infty}\mathbf{E}\left(\omega_{j,t}\right) = \omega_{j,0}$. □

This proposition suggests that centralization is unlikely to occur in the validator space, as their stake shares follow a martingale process and depend only on their initial shares. We next analyze the evolution of the total stake controlled by the validators in the system.

Lemma 4.2. *Under Assumption 2.10, the sequence $\{S_t\}_{t\geq 1}$ is strictly increasing.*

Proof. We prove the lemma by induction. At $t = 1$, we have

$$S_1 = S_0 + R_1 - \frac{\alpha}{S_0^{\gamma}}.$$

By Assumption 2.10, it follows that $\frac{\alpha}{S_0^{\gamma}} < \min\{k_1, k_2, \ldots, k_M\}$ where $k_i = \max\{\beta_v, \mu h_i\}$ for each $i \in [M]$. Since R_1 takes values in $\{k_i\}_{i\in[M]}$, we conclude that $\frac{\alpha}{S_0^{\gamma}} < \min\{k_1, k_2, \ldots, k_M\} \leq R_1$, which implies $S_1 > S_0$. Suppose that $S_0 < S_1 < \ldots < S_k$ holds for some $k > 1$. At $t = k+1$,

$$S_{k+1} = S_k + R_k - \frac{\alpha}{S_k^{\gamma}}.$$

Similarly, $\frac{\alpha}{S_k^{\gamma}} < \left(\frac{S_0}{S_k}\right)^{\gamma} \cdot \min\{k_1, k_2, \ldots, k_M\} < R_k$ since $S_0 < S_k$. Thus, $S_{k+1} > S_k$. This completes the induction and proves the lemma. □

Proposition 4.3. *(Long-time behavior of S_t) The following results hold:*

(1) The process $(S_t, t \geq 0)$ is an $\mathcal{F}_t$-sub-martingale, and its compensator is

$$A_t = Rt - \alpha\sum_{k=0}^{t-1}\frac{1}{S_k^{\gamma}}.$$

(2) There is the convergence in probability:

  (a) If $\gamma > 0$, then

$$\frac{S_t}{t} \to R \quad \text{as } t \to \infty.$$

(b) If $\gamma = 0$, then

$$\frac{S_t}{t} \to R - \alpha \quad \text{as } t \to \infty.$$

Proof. (1) It suffices to note that $\mathbf{E}\left(S_{t+1} | \mathcal{F}_t\right) = S_t + R - \frac{\alpha}{S_t^\gamma}$.

(2) Apply the method of moments by computing $\mathbf{E}\left(S_t^k\right)$ for all k. For $k = 1$, we have by definition:

$$\begin{aligned}\mathbf{E}\left(S_{t+1} - S_t | S_t = s\right) &= R - \frac{\alpha}{s^\gamma}, \\ &= \begin{cases} R & \text{if } \gamma > 0 \\ R - \alpha & \text{if } \gamma = 0 \end{cases} \quad \text{as } s \to \infty.\end{aligned}$$

It is clear that with probability one $S_t \to \infty$ as $t \to \infty$ when $\gamma \geq 0$. As a result,

$$\begin{cases} \mathbf{E}\left(S_{t+1} - S_t\right) \to R \quad \text{as } t \to \infty & \text{if } \gamma > 0, \\ \mathbf{E}\left(S_{t+1} - S_t\right) \to R - \alpha \quad \text{as } t \to \infty & \text{if } \gamma = 0. \end{cases}$$

which yields

$$\begin{cases} \mathbf{E}\left(S_t\right) \sim Rt \quad \text{as } t \to \infty & \text{if } \gamma > 0, \\ \mathbf{E}\left(S_t\right) \sim \left(R - \alpha\right) t \quad \text{as } t \to \infty & \text{if } \gamma = 0. \end{cases}$$

Next for $k = 2$, we have:

$$\begin{aligned}\mathbf{E}\left(S_{t+1}^2 - S_t^2 | S_t = s\right) &= \mathbf{E}\left(R_{t+1}^2\right) + \frac{\alpha^2}{s^{2\gamma}} + 2sR - 2\frac{\alpha s}{s^\gamma} - 2\frac{\alpha}{s^\gamma}R, \\ &= \begin{cases} \mathbf{E}\left(R_{t+1}^2\right) + 2Rs + \mathcal{O}(s^{1-\gamma}) & \text{if } \gamma > 0 \\ \mathbf{E}\left(R_{t+1}^2\right) + \alpha^2 + 2\left(R - \alpha\right) s - 2\alpha R & \text{if } \gamma = 0 \end{cases} \quad \text{as } s \to \infty.\end{aligned}$$

Thus,

$$\begin{cases} \mathbf{E}\left(S_{t+1}^2 - S_t^2\right) = \left(2R + o(1)\right) \mathbf{E}\left(S_t\right) \sim 2R^2 t & \text{if } \gamma > 0, \\ \mathbf{E}\left(S_{t+1}^2 - S_t^2\right) = \left(2\left(R - \alpha\right) + o(1)\right) \mathbf{E}\left(S_t\right) \sim 2\left(R - \alpha\right)^2 t & \text{if } \gamma = 0. \end{cases}$$

Then we get:

$$\begin{cases} \mathbf{E}\left(S_t^2\right) \sim R^2 t^2 & \text{if } \gamma > 0 \\ \mathbf{E}\left(S_t^2\right) \sim \left(R - \alpha\right)^2 t^2 & \text{if } \gamma = 0 \end{cases} \quad \text{as } t \to \infty.$$

We proceed by induction. Assume that

$$\begin{cases} \mathbf{E}\left(S_t^k\right) \sim R^k t^k & \text{if } \gamma > 0 \\ \mathbf{E}\left(S_t^k\right) \sim \left(R - \alpha\right)^k t^k & \text{if } \gamma = 0 \end{cases} \quad \text{as } t \to \infty.$$

We obtain:

$$\begin{cases} \mathbf{E}\left(S_{t+1}^{k+1} - S_t^{k+1}\right) = \left(\left(k+1\right) R + o(1)\right) \mathbf{E}\left(S_t^k\right) \sim \left(k+1\right) R^{k+1} t^k & \text{if } \gamma > 0, \\ \mathbf{E}\left(S_{t+1}^{k+1} - S_t^{k+1}\right) = \left(\left(k+1\right)\left(R - \alpha\right) + o(1)\right) \mathbf{E}\left(S_t^k\right) \sim \left(k+1\right)\left(R - \alpha\right)^{k+1} t^k & \text{if } \gamma = 0. \end{cases}$$

Thus, we have:

$$\begin{cases} \mathbf{E}\left(S_t^{k+1}\right) \sim R^{k+1} t^{k+1} \quad \text{as } t \to \infty & \text{if } \gamma > 0, \\ \mathbf{E}\left(S_t^{k+1}\right) \sim \left(R - \alpha\right)^{k+1} t^{k+1} \quad \text{as } t \to \infty & \text{if } \gamma = 0. \end{cases}$$

By the method of moments, S_t/t converges in distribution, and thus in probability to R if $\gamma > 0$, and to $R - \alpha$ if $\gamma = 0$. $\square$

In general, it is difficult to make explicit the distribution of $\omega_{j,\infty}$. Nevertheless, in the case where $\alpha = 0$, the following proposition characterizes the distribution of $\omega_{j,\infty}$.

Recall from Section 2.2 that $\{R_t\}_{t\geq 1}$ is a sequence of i.i.d random variables, where each R_t takes the value

$$k_i := \max\{\mu h_i, \beta_v\},$$

with probability

$$p_i := \frac{h_i}{\sum_{k=1}^{M} h_k} \quad \text{for } i \in [M].$$

Let $\mathcal{K} := \{k_i\}_{i\in[M]}$ denote the support of R_t for all $t \geq 1$. Let $\mathscr{P}([0,1])$ denote the space of distribution functions with support in $[0,1]$, and $\mathcal{S} := [0,\infty) \times [0,\infty) \setminus \{(0,0)\}$. For any $x \in \mathbb{R}$, let δ_x be the distribution of the point mass at x. Define the function

$$\mathbf{F}_j : \mathcal{S} \to \mathscr{P}([0,1]),$$

that maps the initial stake pair $(s_{j,0}, S_0 - s_{j,0})$, where $S_0 = \sum_{j=1}^{N} s_{j,0}$, to the probability distribution $\mathbf{F}_j(s_{j,0}, S_0 - s_{j,0})$ of the limiting stake share $\omega_{j,\infty}$. Recall that the initial stake share is given by $\omega_{j,0} = \frac{s_{j,0}}{S_0}$.

Proposition 4.4. *When $\alpha = 0$, for all $(s_{j,0}, S_0 - s_{j,0}) \in \mathcal{S}$, the distribution function of $\omega_{j,\infty}$ satisfies*

$$\mathbf{F}_j(s_{j,0}, S_0 - s_{j,0}) = \omega_{j,0} \sum_{k_i\in\mathcal{K}} \mathbf{F}_j(s_{j,0} + k_i, S_0 - s_{j,0})\, p_i + (1-\omega_{j,0}) \sum_{k_i\in\mathcal{K}} \mathbf{F}_j(s_{j,0}, S_0 - s_{j,0} + k_i)\, p_i, \tag{4.3}$$

and it is the unique solution to Eq. (4.3) among the continuous functions $\mathscr{G} : \mathcal{S} \to \mathscr{P}([0,1])$ satisfying the following three conditions:

(1) $\mathscr{G}(0,a) = \delta_0$ for $a > 0$;
(2) $\mathscr{G}(a,0) = \delta_1$ for $a > 0$;
(3) For every $\epsilon > 0$, there exists a $C = C(\epsilon)$ such that

$$d_w\left(\mathscr{G}(s_{j,0}, S_0 - s_{j,0}), \delta_{\omega_{j,0}}\right) < \epsilon,$$

if $S_0 > C$, where

$$d_w(F,G) = \int_0^1 |F(x) - G(x)|dx \quad \text{for all } F, G \in \mathscr{P}([0,1]).$$

Proof. When $\alpha = 0$, the stake share evolution of each validator j with initial stake $s_{j,0}$ can be modeled as a Pólya urn model with random replacements. Specifically, we consider an urn containing two types of balls, black and white, where the initial number of black balls is $s_{j,0}$ and the initial number of white balls is $S_0 - s_{j,0}$. The remainder of the proof follows Aletti et al. [1]. $\square$

Refer to Section 5.2 for the numerical results on the distribution of $\omega_{j,\infty}$. Comparing Figure 3 and Figure 4, we observe that when $\gamma > 0$, the distribution of $\omega_{j,\infty}$ appears to be very similar for $\alpha = 0$ and $\alpha \neq 0$. To provide an explanation, recall that when $\alpha = 0$, the evolution of $\omega_{j,t}$ is given by:

$$\begin{aligned}\omega_{j,t+1} &= \omega_{j,t}\frac{S_t - \alpha/S_t^{\gamma}}{S_{t+1}} + \frac{R_{t+1}}{S_{t+1}}\mathbf{1}\left(X_{j,t+1}\right),\\ &= \omega_{j,t}\frac{S_t}{S_{t+1}} + \frac{R_{t+1}}{S_{t+1}}\mathbf{1}\left(X_{j,t+1}\right).\end{aligned}$$

For $\gamma > 0$, $S_t \to \infty$ as shown in Proposition 4.3. Since $\omega_{j,t}$ is bounded, the term $\alpha\omega_{j,t}/S_t^{\gamma}$ in (4.1) vanishes asymptotically. As a result, the evolution of stake shares follows a similar pattern to the case when $\alpha = 0$.

When $\gamma = 0$, we can compute explicitly the variance of $\omega_{j,\infty}$, and study its stability. The following theorem characterizes its asymptotics.

Theorem 4.5. *Let $R_{\min}$ and $R_{\max}$ denote the minimum and maximum values, respectively, of the i.i.d random variable R_t:*

$$R_{\min} = \min_{i\in[M]}\{k_i\}, \quad R_{\max} = \max_{i\in[M]}\{k_i\}.$$

For $s_{j,0} = f(S_0)$ such that $f(S_0) \to \infty$ as $S_0 \to \infty$, we have for each $\epsilon > 0$ and each $t \geq 1$ or $t = \infty$:

$$\mathbf{P}\left(\left|\frac{\omega_{j,t}}{\omega_{j,0}} - 1\right| > \epsilon\right) \leq \frac{R_{\max}^2}{\left(R_{\min} - \alpha\right)\epsilon^2 f\left(S_0\right)}, \tag{4.4}$$

which converges to 0, as $S_0 \to \infty$.

The proof of Theorem 4.5 is given in Appendix B. It implies that for large validators, i.e., those with initial stakes $s_{j,0} = f(S_0)$ such that $f(S_0) \to \infty$ as $S_0 \to \infty$, their shares remain stable over time. Specifically, their limiting share converges in probability to their initial share as the total initial stake $S_0 \to \infty$.

5. Numerical Results

This section presents numerical results on the strategic interactions among builders and the evolution of validators' stake shares. Section 5.1 reports numerical findings for the multi-player setting ($M \geq 3$) in the builders' game, and Section 5.2 provides simulation results on the dynamics of validators' stake shares.

5.1. **Multi-Player Game among Builders** ($M \geq 3$).

We conduct numerical experiments to analyze the equilibrium behavior in settings with more than two players. In this analysis, we consider two cases:

(1) Each player's ability to extract MEV without the order being auctioned in the OFA is $k_{1,i} := \frac{\bar{v}_i}{\bar{f}_i}$imes the MEV obtained from the auctioned order, and this ratio $k_{1,i}$emains constant across all three players, i.e., $k_{1,1} = k_{1,2} = k_{1,3}$
(2) The ratio $k_{1,i}$aries among the three players, such that $k_{1,1} < k_{1,2} < k_{1,3}$

The results for three players are presented in Table 2 and Table 3.

From Table 2, we observe that, in most cases, the results align with the trend in Proposition 3.7: the more capable players pay relatively less. However, an exception occurs when $\bar{f}_i/\bar{v}_i = 2$ for $i = 1, 2, 3$, $\bar{f}_2/\bar{f}_3 = \bar{v}_2/\bar{v}_3 = 2$ and $h_2/h_3 > 2$. This suggests that the presence of

$\bar{f}_i/\bar{v}_i$	2	3	5	8	10
h_1	138.55	184.96	278.07	417.99	511.34
$\mathbf{E}(\pi_1)$	147.81	203.40	315.73	484.98	597.97
h_2	64.31	89.42	139.38	214.14	263.93
$\mathbf{E}(\pi_2)$	19.79	30.01	50.48	81.21	101.70
h_3	31.58	45.50	73.07	114.27	141.69
$\mathbf{E}(\pi_3)$	13.00	20.34	35.14	57.40	72.25

TABLE 2. This table presents the equilibrium outcomes for three players under the assumption that each player $i \in [3]$ has a common ratio $\bar{f}_i/\bar{v}_i$, where the ratio is set to $2, 3, 5, 8$, or 10. The players' $\bar{v}_i$ follow the ratio $\bar{v}_1 : \bar{v}_2 : \bar{v}_3 = 5 : 2 : 1$, with $\bar{v}_3 = 30$.

$\bar{v}_3$	10	20	30	50	80
h_1	6099.31	12198.62	18297.94	30496.56	48794.50
$\mathbf{E}(\pi_1)$	37850.41	75700.83	113551.24	189252.07	302803.32
h_2	932.00	1864.01	2796.01	4660.01	7456.02
$\mathbf{E}(\pi_2)$	140.91	281.82	422.74	704.56	1127.30
h_3	49.89	99.79	149.68	249.47	399.15
$\mathbf{E}(\pi_3)$	13.74	27.48	41.22	68.71	109.93

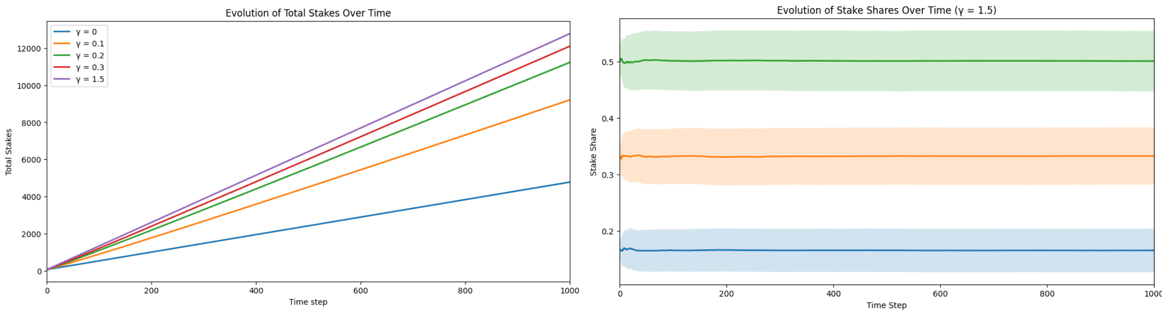
TABLE 3. This table presents the equilibrium outcomes for three players under the assumption that $\frac{\bar{f}_1}{\bar{v}_1} = 1000 > \frac{\bar{f}_2}{\bar{v}_2} = 100 > \frac{\bar{f}_3}{\bar{v}_3} = 10$. The players' $\bar{v}_i$ follow the ratio $\bar{v}_1 : \bar{v}_2 : \bar{v}_3 = 5 : 2 : 1$, where $\bar{v}_3$ is set to $10, 20, 30, 50$, or 80.

player 3 may exert competitive pressure on player 2, leading to a higher payment. Furthermore, the most capable player appears to gain a disproportionately higher expected utility, while the gap in expected utility between the second and least capable players narrows.

From Table 3, we observe that when every player in the game is scaled by k times, then their equilibrium solutions and expected utilities are also scaled by k times. The effect observed in Table 2 is further amplified here. More capable players, such as player 1, attain disproportionately high expected utility, significantly exceeding $\frac{\bar{f}_1\bar{v}_1}{\bar{f}_2\bar{v}_2}$.

5.2. **Simulation of Validators' Stake Shares.** We simulate the evolution of total stake in the PoS system, and the distribution of validators' stake shares. The parameters are set as follows: the number of validators is fixed at $N = 3$, with a time horizon of $T = 1000$ steps. The initial stakes of the three validators are given by $s_{1,0} = 10, s_{2,0} = 20$, and $s_{3,0} = 30$, respectively. There are two builders, with bids set at $h_1 = 15$ and $h_2 = 20$. Additionally, we set $\mu = 0.7$, $\alpha = 8$, and $\beta_v = 11$, while γ varies over the values $0, 0.1, 0.2, 0.3$, and 1.5. We run the simulation 1000 times and plot the mean total stake and mean stake shares over time in Figure 2. Additionally, we repeat the simulation 10000 times and plot the distribution of the final stake share, $\omega_{j,T}$, for $j = 1, 2, 3$, in Figure 3. We then set $\alpha = 0$ and plot the distribution of $\omega_{j,T}$ in Figure 4.

From Figure 2, we observe that the total stake grows at a rate of R when $\gamma > 0$ and at a rate of $R - \alpha$ when $\gamma = 1.5$. The slight variations in growth rates for $\gamma = 0.1, 0.2, 0.3$ in the



(a) (b)

FIGURE 2. (a) Evolution of mean total stakes over time when $\gamma = 0, 0.1, 0.2, 0.3,$ and 1.5. (b) Evolution of mean stake shares over time when $\gamma = 1.5$. The shaded region represents mean $\pm\,\frac{1}{4}$standard deviation.

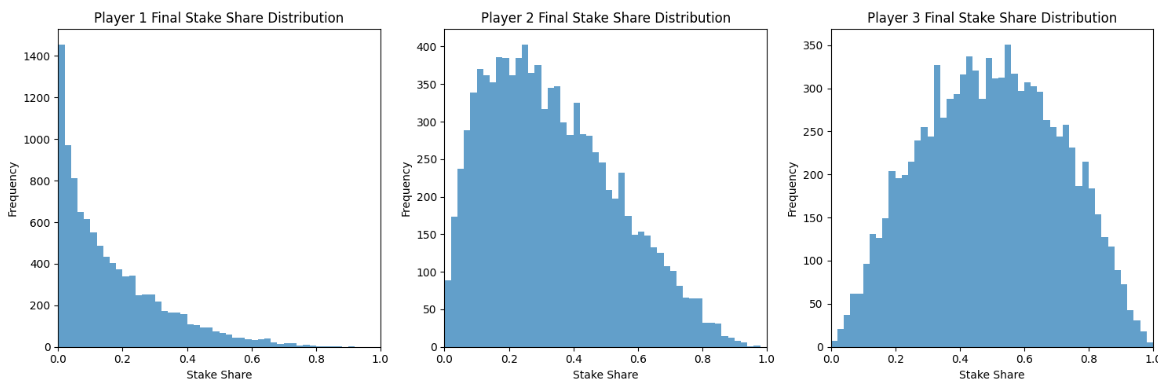


FIGURE 3. Distribution of final stake shares $(\alpha \neq 0, \gamma = 1.5)$

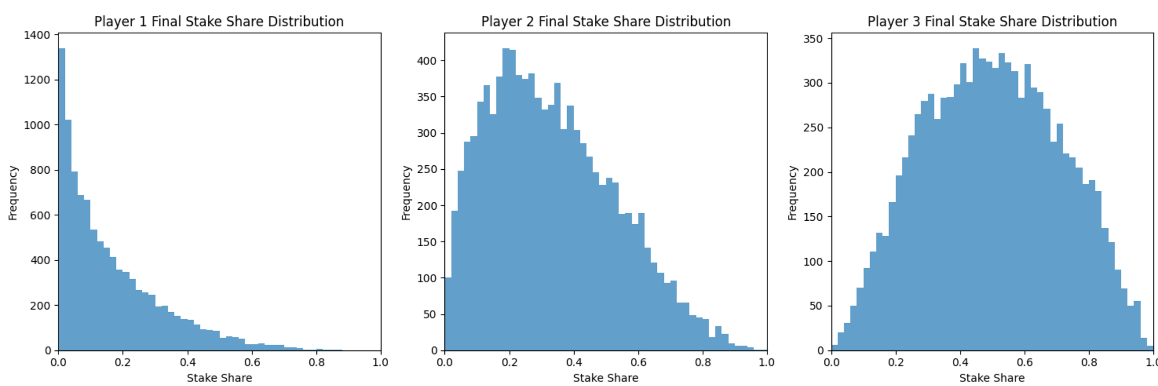


FIGURE 4. Distribution of final stake shares $(\alpha = 0, \gamma = 1.5)$

plot arise from the fact that $1/S_t^\gamma$ decays at different rates as $S_t \to \infty$. Extending the time horizon would further demonstrate that the growth rate converges to $R - \alpha$.

For the evolution of stake shares, Figure 2 shows that they remain stable over time, consistent with their martingale property. Since the stake share dynamics exhibit the same behavior across different values of γ, we present only a representative case in the figure.

Figure 3 and Figure 4 illustrate the marginal distribution of $\omega_{j,T}$ for $j = 1, 2, 3$ in the cases where $\alpha > 0$ and $\alpha = 0$, respectively. Each player begins with initial stakes $10, 20$, and 30, corresponding to initial stake shares of $\omega_{1,0} = \frac{1}{6}, \omega_{2,0} = \frac{1}{3}$, and $\omega_{3,0} = \frac{1}{2}$. The distributions in both cases appear highly similar, with their marginal distributions skewed toward their initial stake shares.

When increasing the total initial stake while maintaining the initial relative shares, we observe that the limiting stake shares indeed converge to their initial values, aligning with the theoretical results established in Theorem 4.5.

However, when $\gamma = 0$, we observe that under high consumption costs and small initial stakes, some validators' stake shares are likely to be driven to zero, as shown in Figures 5 and 6 in Appendix C. This phenomenon may be attributed to the fact that costs impose a greater relative burden on smaller validators, whereas larger validators experience only a limited impact. Specifically, since the term $\alpha s_{j,0}/S_0$ is bounded, larger validators are more resilient to costs, while smaller validators are more vulnerable and face greater challenges in accumulating stakes over time.

6. Conclusions

In this paper, we examine the interaction between the order flow auction and the block-building auction, formulating the problem within a general multiplayer framework. For the case of two players, we establish the existence and uniqueness of the Nash equilibrium, while for more than two players, we conduct simulations. Both analyses suggest a tendency toward centralization in the builder space. In contrast, we find that validators' stake shares follow a martingale process, and both theoretical and numerical results indicate that centralization is unlikely to emerge in the validator space.

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Appendix

Appendix A. Code for the Problem 3.8

```python
import numpy as np
from scipy.optimize import minimize

def func(vars):
    f1, f2, v1, v2 = vars
    try:
        p = -((11*f1**2 + 12*v1**2 + 4*f1*(4*f2 + v1 + 8*v2))/(8*f1**2))
        q = (3*f1**3 + 8*v1**3 + 4*f1*v1*(4*f2 + v1 + 8*v2) + f1**2*(-10*v1 + 32*v2))/(8*f1**3)
        delta0 = (2*f1-2*f2+4*v1-4*v2)**2+3*(3*f1+2*v1)*(3*f2+2*v2)-12*f1*f2
        delta1 = -27*f2*(3*f1 + 2*v1)**2 + 72*f1*f2*(2*f1 - 2*f2 + 4*v1 - 4*v2) + 2*(2*f1 - 2*f2 + 4*v1 - 4*v2)**3 - \
                  9*(3*f1 + 2*v1)*(2*f1 - 2*f2 + 4*v1 - 4*v2)*(-3*f2 - 2*v2) + 27*f1*(-3*f2 - 2*v2)**2

        phi = np.arccos(delta1/(2*np.sqrt(delta0**3)))

        S = 1/2*np.sqrt(-2*p/3 + 2/(3*f1)*np.sqrt(delta0)*np.cos(phi/3))

        lmda = -(3*f1+2*v1)/(4*f1)+S+1/2*np.sqrt(-4*S**2-2*p-q/S)
        h1_1 = lmda*(f1*lmda+f1+2*v1)/((1+2*lmda)*(1+lmda))
        h1_2 = (f2 + (f2+2*v2)*lmda)/(lmda*(2+lmda)*(1+lmda))

        return (8*S**3-q)
    except:
        return np.inf

initial_guess = [1, 1, 1, 1]

# Variable bounds (all variables > 0)
bounds = [(1e-5, None), (1e-5, None), (1e-5, None), (1e-5, None)]

# Define the constraints f1 >= v1 and f2 >= v2
constraints = [
    {"type": "ineq", "fun": lambda vars: vars[0] - vars[2]},  # f1 - v1 >= 0
    {"type": "ineq", "fun": lambda vars: vars[1] - vars[3]},  # f2 - v2 >= 0
]

result = minimize(func, initial_guess, method='SLSQP', bounds=bounds, constraints=constraints)

# Results
if result.success:
    print("Global minimum found:")
    print("Function value:", result.fun)
    print("At variables (f1, f2, v1, v2):", result.x)
else:
    print("Optimization failed:", result.message)
```

Appendix B. Proof of Theorem 4.5

To prove Theorem 4.5, we need a series of lemmas.

Lemma B.1. *Let $Var_t(\cdot)$ and $\mathbf{E}_t(\cdot)$ denote the conditional variance and conditional expectation at time t, respectively, i.e., $Var_t(\cdot) = Var(\cdot \mid \mathcal{F}_t)$, $\mathbf{E}_t(\cdot) = \mathbf{E}(\cdot \mid \mathcal{F}_t)$. When $\gamma = 0$, the conditional variance at time t of validator j's share at time $t+1$ is:*

$$Var_t(\omega_{j,t+1}) = \omega_{j,t}(1-\omega_{j,t})\mathbf{E}_t\left[\left(\frac{R_{t+1}}{S_{t+1}}\right)^2\right].$$

Proof. When $\gamma = 0$,

$$\omega_{j,t+1} = \omega_{j,t}\frac{S_t-\alpha}{S_{t+1}} + \frac{R_{t+1}}{S_{t+1}}\mathbf{1}(X_{j,t+1}).$$

Noting that $\frac{S_t-\alpha}{S_{t+1}} = 1 - \frac{R_{t+1}}{S_{t+1}}$, we have

$$\begin{aligned}\mathbf{E}_t\left(\omega_{j,t+1}^2\right) &= \mathbf{E}_t\left[\omega_{j,t}^2\left(1-\frac{R_{t+1}}{S_{t+1}}\right)^2 + \left(\frac{R_{t+1}}{S_{t+1}}\right)^2\mathbf{1}(X_{j,t+1}) + 2\omega_{j,t}\mathbf{1}(X_{j,t+1})\frac{R_{t+1}}{S_{t+1}}\left(1-\frac{R_{t+1}}{S_{t+1}}\right)\right],\\ &= \omega_{j,t}^2 + \omega_{j,t}(1-\omega_{j,t})\mathbf{E}_t\left[\left(\frac{R_{t+1}}{S_{t+1}}\right)^2\right].\end{aligned}$$

where the last equality is obtained by Assumption 2.11. By Theorem 4.1, $(\omega_{j,t})_{t\geq 0}$ is a martingale:

$$\mathbf{E}_t(\omega_{j,t+1}) = \omega_{j,t}.$$

Therefore, the conditional variance of $\omega_{j,t+1}$ is given by

$$\begin{aligned}\mathrm{Var}_t(\omega_{j,t+1}) &= \mathbf{E}_t\left(\omega_{j,t+1}^2\right) - \left(\mathbf{E}_t(\omega_{j,t+1})\right)^2,\\ &= \omega_{j,t}(1-\omega_{j,t})\mathbf{E}_t\left[\left(\frac{R_{t+1}}{S_{t+1}}\right)^2\right].\end{aligned}$$

□

Lemma B.2. *When $\gamma = 0$, the unconditional variance of validator j's share at time t is given by*

$$Var(\omega_{j,t+1}) = a_t\omega_{j,0}(1-\omega_{j,0}), \tag{B.1}$$

where the sequence a_t satisfies

$$a_1 = z_1, \quad a_{t+1} = a_t + z_{t+1}(1-a_t), \tag{B.2}$$

where

$$z_{t+1} = \mathbf{E}\left[\left(\frac{R_{t+1}}{S_{t+1}}\right)^2\right]. \tag{B.3}$$

Proof. We prove by induction, following the approach of Rosu and Saleh [18]. Since $\mathbf{E}_t(\omega_{j,t+1}) = \omega_{j,t}$, the following equation holds for all $t \geq 0$:

$$\mathrm{Var}(\omega_{j,t+1}) = \mathrm{Var}(\omega_{j,t}) + \mathbf{E}(\mathrm{Var}_t(\omega_{j,t+1})). \tag{B.4}$$

From Lemma B.1, we establish the base case at $t = 1$:

$$\begin{aligned}\mathrm{Var}\left(\omega_{j,1}\right) &= \omega_{j,0}\left(1-\omega_{j,0}\right)\mathbf{E}\left[\left(\frac{R_1}{S_1}\right)^2\right],\\ &= z_1\omega_{j,0}\left(1-\omega_{j,0}\right).\end{aligned}$$

Next, we assume that Eq. (B.1) holds for all time steps up to $t = k$, where $k > 1$. At time $t = k + 1$, we have

$$\mathrm{Var}\left(\omega_{j,t+1}\right) = \mathrm{Var}\left(\omega_{j,t}\right) + \mathbf{E}\left\{\omega_{j,t}\left(1-\omega_{j,t}\right)\mathbf{E}_t\left[\left(\frac{R_{t+1}}{S_{t+1}}\right)^2\right]\right\},$$

from Lemma B.1 and Eq. (B.4). Since $\mathrm{Var}\left(\omega_{j,t}\right) = a_t\omega_{j,0}\left(1-\omega_{j,0}\right)$, we have

$$\begin{aligned}\mathrm{Var}\left(\omega_{j,t+1}\right) &= a_t\omega_{j,0}\left(1-\omega_{j,0}\right) + \left(\omega_{j,0} - a_t\omega_{j,0}\left(1-\omega_{j,0}\right) - \omega_{j,0}^2\right)\cdot\mathbf{E}\left[\left(\frac{R_{t+1}}{S_{t+1}}\right)^2\right],\\ &= \omega_{j,0}\left(1-\omega_{j,0}\right)\left\{a_t + \left(1-a_t\right)\mathbf{E}\left[\left(\frac{R_{t+1}}{S_{t+1}}\right)^2\right]\right\}.\end{aligned}$$

This completes the induction step and proves the lemma. □

Lemma B.3. *When $\gamma = 0$, let the sequence $\{a_t\}_{t\geq 0}$ be defined as in Eqs. (B.2)–(B.3) and extend it by setting $a_0 = 0$. This sequence satisfies $a_t \in [0, 1]$ and is non-decreasing, i.e., $a_{t+1} \geq a_t$, for all t.*

Proof. We prove by induction. When $\gamma = 0$, the sequence $\{S_t\}$ satisfies the recurrence relation

$$S_{t+1} = S_t + R_{t+1} - \alpha$$

as given by Eq. (2.3). For the base case $t = 1$, by Eqs. (B.2)–(B.3), we have

$$a_1 = \mathbf{E}\left[\left(\frac{R_1}{S_1}\right)^2\right] = \mathbf{E}\left[\left(\frac{R_1}{S_0 + R_1 - \alpha}\right)^2\right] \in [0,1],$$

since $\alpha < S_0$ by Assumption 2.10. Clearly, $a_1 \geq a_0 = 0$. Suppose that $a_t \in [0, 1]$ and $a_{t+1} \geq a_t$, for all $1 \leq t \leq k$. At $t = k + 1$, we have $a_{k+1} = a_k + z_{k+1}(1 - a_k)$. By the definition of z_t,

$$\begin{aligned}z_{k+1} &= \mathbf{E}\left[\left(\frac{R_{k+1}}{S_{k+1}}\right)^2\right],\\ &= \mathbf{E}\left[\left(\frac{R_{k+1}}{S_0 + \sum_{i=1}^{k+1} R_i - (k+1)\,\alpha}\right)^2\right] \in [0,1],\end{aligned}$$

since $\alpha < S_0$ and $\alpha < R_{\min}$ by Assumption 2.10. Therefore, it follows that $a_{k+1} \geq a_k$ and $a_{k+1} \in [0, 1]$. This completes the induction step and the lemma is proved. □

Lemma B.4. *Let a_t be defined by Eqs. (B.2)–(B.3). We have for each $t \geq 1$,*

$$a_t \leq \frac{R_{\max}^2}{S_0\left(R_{\min}-\alpha\right)}. \tag{B.5}$$

Proof. By Eqs. (B.2)–(B.3), we have

$$a_t - a_0 = \sum_{i=1}^{t} z_i\left(1-a_{i-1}\right),$$

noting that $a_0 := 0$. It follows from Lemma B.3 that $a_t \in [0,1]$. Consequently, we obtain

$$\begin{aligned} a_t &\leq \sum_{i=1}^{t} z_i = \sum_{i=1}^{t} \mathbf{E}\left[\left(\frac{R_i}{S_i}\right)^2\right], \\ &\leq \sum_{i=1}^{t}\left(\frac{R_{\max}}{S_0+i\left(R_{\min}-\alpha\right)}\right)^2, \end{aligned}$$

where the last inequality follows from the fact that $S_i = S_0 + \sum_{j=1}^{i} R_i - i\alpha$. Since the function $x \to \left(\frac{R_{\max}}{S_0+x(R_{\min}-\alpha)}\right)^2$ is decreasing on $\mathbb{R}_+$ due to $R_{\max} > 0, S_0 > 0$, and $R_{\min} - \alpha > 0$ (by Assumption 2.10), using the sum-integral trick as in Rosu and Saleh [18, Lemma A.4], we obtain:

$$a_t \leq \int_0^{\infty}\left(\frac{R_{\max}}{S_0+x\left(R_{\min}-\alpha\right)}\right)^2 dx = \frac{R_{\max}^2}{S_0\left(R_{\min}-\alpha\right)}. \tag{B.6}$$

□

Proof of Theorem 4.5. By Lemma B.2, Chebyshev's inequality, and the uppder bound in B.5, we get

$$\mathbf{P}\left(\left|\frac{\omega_{j,t}}{\omega_{j,0}}-1\right| > \epsilon\right) \leq \frac{a_t\left(1-\omega_{j,0}\right)}{\epsilon^2 \omega_{j,0}} \leq \frac{R_{\max}^2}{\epsilon^2 s_{j,0}\left(R_{\min}-\alpha\right)},$$

since $S_0\omega_{j,0} = s_{j,0}$ and $0 \leq 1-\omega_{j,0} \leq 1$. This proves the estimate (4.4).

Appendix C. Additional Results for Numerical Experiments

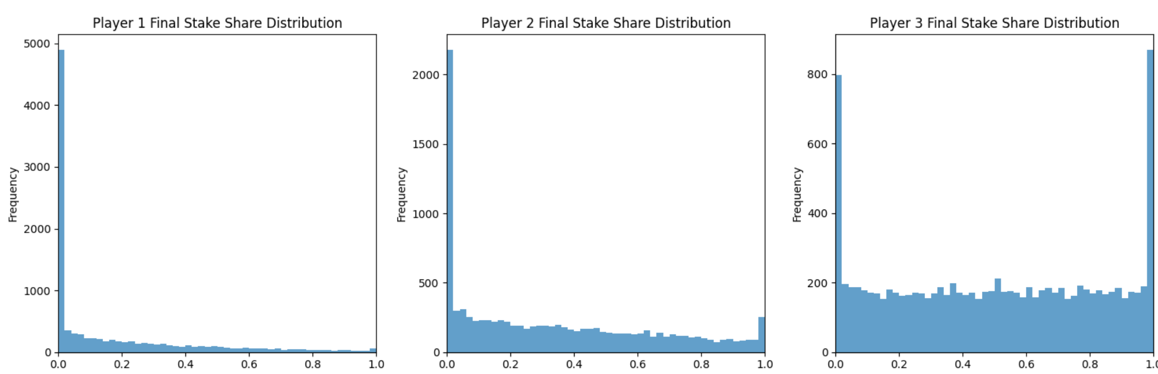

Figure 5. Distribution of final stake shares ($\alpha \neq 0, \gamma = 0$)

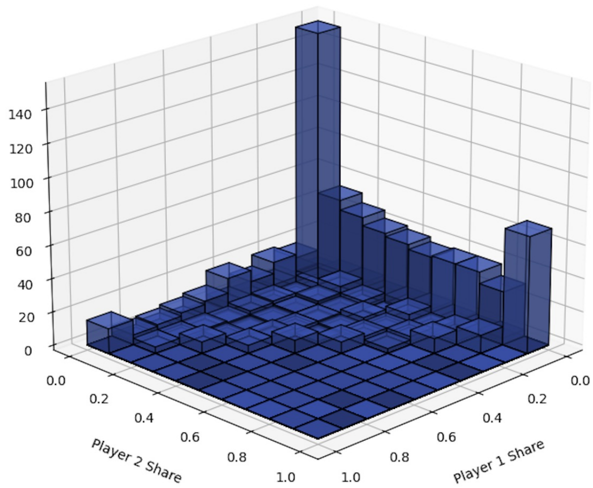

FIGURE 6. Histogram of the joint distribution of final stake shares ($\alpha \neq 0, \gamma = 0$)

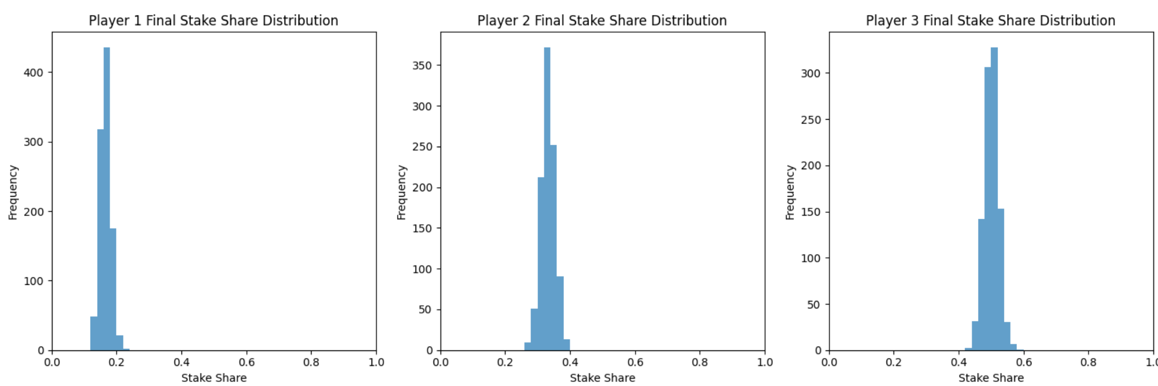

FIGURE 7. Distribution of final stake shares with large initial stakes ($\alpha \neq 0, \gamma = 1.5$)

Figure 5 illustrates the limiting stake distribution under conditions of high consumption costs and small initial stakes, specifically with $\gamma = 0, \alpha = 8$, and initial stakes $s_{1,0} = 10, s_{2,0} = 20, s_{3,0} = 30$. The reward R_t follows a similar setting as in previous cases. Under this setting, we observe that some validators' stake shares are likely to be driven toward zero, as explained in Section 5.2. Their joint distribution is shown in Figure 6.

Compared to the previous setting, where the initial stakes for the three players are $10, 20$, and 30, as shown in Figures 3, 4, and 5, we set the initial stakes to 1000, 2000, and 3000, respectively, in Figure 7. In this setting, we observe that the limiting share, $\omega_{j,T}$, indeed converges to the initial share, consistent with the theoretical results established in Theorem 4.5.

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