Patterns in Random Walks and Brownian Motion

Jim Pitman and Wenpin Tang

Abstract We ask if it is possible to find some particular continuous paths of unit length in linear Brownian motion. Beginning with a discrete version of the problem, we derive the asymptotics of the expected waiting time for several interesting patterns. These suggest corresponding results on the existence/non-existence of continuous paths embedded in Brownian motion. With further effort we are able to prove some of these existence and non-existence results by various stochastic analysis arguments. A list of open problems is presented.

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1 Introduction and Main Results

We are interested in the question of embedding some continuous-time stochastic processes $(Z_u, 0 \leq u \leq 1)$ into a Brownian path $(B_t; t \geq 0)$, without time-change or scaling, just by a random translation of origin in spacetime. More precisely, we ask the following:

Question 1 Given some distribution of a process Z with continuous paths, does there exist a random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as $(Z_u, 0 \leq u \leq 1)$?

The question of whether external randomization is allowed to construct such a random time T, is of no importance here. In fact, we can simply ignore Brownian

J. Pitman (✉) • W. Tang
Department of Statistics, University of California, 367 Evans Hall, Berkeley, CA 94720-3860, USA
e-mail: pitman@stat.berkeley.edu; wenpintang@stat.berkeley.edu

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motion on $[0,1]$, and consider only random times $T \geq 1$. Then $(B_t; 0 \leq t \leq 1)$ provides an independent random element which is adequate for any randomization, see e.g. Kallenberg [40, Theorem 6.10].

Note that a continuous-time process whose sample paths have different regularity, e.g. fractional Brownian motion with Hurst parameter $H \neq \frac{1}{2}$, cannot be embedded into Brownian motion. Given $(B_t; t \geq 0)$ standard Brownian motion, we define $g_1 := \sup\{t < 1; B_t = 0\}$ the time of last exit from 0 before $t = 1$, and $d_1 := \inf\{t > 1; B_t = 0\}$ the first hitting time of 0 after $t = 1$. The following processes, derived from Brownian motion, are of special interest.

- Brownian bridge, which can be defined by

$$\left(b_u^0 := \frac{1}{\sqrt{g_1}} B_{ug_1}; 0 \leq u \leq 1\right),$$

  and its reflected counterpart $(|b_u^0|; 0 \leq u \leq 1)$.
- Normalized Brownian excursion defined by

$$\left(e_u := \frac{1}{\sqrt{d_1 - g_1}} |B_{g_1 + u(d_1 - g_1)}|; 0 \leq u \leq 1\right).$$

- Brownian meander defined as

$$\left(m_u := \frac{1}{\sqrt{1 - g_1}} |B_{g_1 + u(1 - g_1)}|; 0 \leq u \leq 1\right).$$

- Brownian co-meander defined as

$$\left(\tilde{m}_u := \frac{1}{\sqrt{d_1 - 1}} |B_{d_1 - u(d_1 - 1)}|; 0 \leq u \leq 1\right).$$

- The three-dimensional Bessel process

$$\left(R_u := \sqrt{(B_u)^2 + (B_u')^2 + (B_u'')^2}; 0 \leq u \leq 1\right),$$

  where $(B_t'; t \geq 0)$ and $(B_u''; u \geq 0)$ are two independent copies of $(B_t; t \geq 0)$.
- The first passage bridge through level $\lambda \neq 0$, defined by

$$(F_u^{\lambda,br}; 0 \leq u \leq 1) \stackrel{(d)}{=} (B_u; 0 \leq u \leq 1) \quad \text{conditioned on} \quad \tau_\lambda = 1,$$

  where $\tau_\lambda := \inf\{t \geq 0; B_t = \lambda\}$ is the first time at which Brownian motion hits $\lambda \neq 0$. Note that for $\lambda < 0$, $(F_u^{|\lambda|,br}; 0 \leq u \leq 1) \stackrel{(d)}{=} (-F_u^{\lambda,br}; 0 \leq u \leq 1)$, and $(F_{1-u}^{\lambda,br} + |\lambda|; 0 \leq u \leq 1)$ is distributed as three dimensional Bessel bridge ending at $|\lambda| > 0$, see e.g. Biane and Yor [10].

- The Vervaat transform of Brownian motion, defined as

$$\left(V_u := \begin{cases} B_{\tau+u} - B_\tau & \text{for } 0 \leq u \leq 1-\tau \\ B_{\tau-1+u} + B_1 - B_\tau & \text{for } 1-\tau \leq u \leq 1 \end{cases} ; 0 \leq u \leq 1 \right),$$

where $\tau := argmin_{0 \leq t \leq 1} B_t$, and the Vervaat transform of Brownian bridge with endpoint $\lambda \in \mathbb{R}$

$$\left(V_u^\lambda := \begin{cases} b^\lambda_{\tau+u} - b^\lambda_\tau & \text{for } 0 \leq u \leq 1-\tau \\ b^\lambda_{\tau-1+u} + \lambda - b^\lambda_\tau & \text{for } 1-\tau \leq u \leq 1 \end{cases} ; 0 \leq u \leq 1 \right),$$

where $(b^\lambda_u; 0 \leq u \leq 1)$ is Brownian bridge ending at $\lambda \in \mathbb{R}$ and $\tau := argmin_{0 \leq t \leq 1} b^\lambda_t$. It was proved by Vervaat [85] that $(V^0_u; 0 \leq u \leq 1) \overset{(d)}{=} (e_u; 0 \leq u \leq 1)$. For $\lambda < 0$, $(V^{|\lambda|}_u; 0 \leq u \leq 1)$ has the same distribution as $(V^\lambda_{1-u} + |\lambda|; 0 \leq u \leq 1)$.

The Brownian bridge, meander, excursion and the three-dimensional Bessel process are well-known. The definition of the co-meander is found in Yen and Yor [91, Chap. 7]. The first passage bridge is studied by Bertoin et al. [8]. The Vervaat transform of Brownian bridges and of Brownian motion are extensively discussed in Lupu et al. [56]. According to the above definitions, the distributions of the Brownian bridge, excursion and (co-)meander can all be achieved in Brownian motion provided some Brownian scaling operation is allowed. Note that the distributions of all these processes are singular with respect to Wiener measure. So it is a non-trivial question whether copies of them can be found in Brownian motion just by a shift of origin in spacetime. Otherwise, for a process $(Z_t, 0 \leq t \leq 1)$ whose distribution is absolutely continuous with respect to that of $(B_t, 0 \leq t \leq 1)$, for instance the Brownian motion with drift $Z_t := \vartheta t + B_t$ for a fixed ϑ, the distribution of Z can be easily obtained as that of $(B_{T+t} - B_T, 0 \leq t \leq 1)$ for a suitable stopping time $T + 1$ by *Rost's filling scheme*. We refer readers to Sect. 3.5 for further development.

The question raised here has some affinity to the question of embedding a given one-dimensional distribution as the distribution of B_T for a random time T. This *Skorokhod embedding problem* traces back to Skorokhod [80] and Dubins [24]– who found integrable stopping times T such that the distribution of B_T coincides with any prescribed one with zero mean and finite second moment. Monroe [64, 65] considered embedding of a continuous-time process into Brownian motion, and showed that every semi-martingale is a time-changed Brownian motion. Rost [76] studied the problem of embedding a one-dimensional distribution in a Markov process with randomized stopping times. We refer readers to the excellent survey of Obloj [69] and references therein. Let $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$ be the moving-window process associated to Brownian motion. In Question 1, we are concerned with the possibility of embedding a given distribution on $\mathcal{C}[0, 1]$ as that of X_T for some random time T.

Let us present the main results of the paper. We start with a list of continuous-time processes that cannot be embedded into Brownian motion by a shift of origin in spacetime.

Theorem 1 (Impossibility of Embedding of Normalized Excursion, Reflected Bridge, Vervaat Transform of Brownian Motion, First Passage Bridge and Vervaat Bridge) *For each of the following five processes $Z := (Z_u; 0 \leq u \leq 1)$, there is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as Z:*

1. *the normalized Brownian excursion $Z = (e_u; 0 \leq u \leq 1)$;*
2. *the reflected Brownian bridge $Z = (|b_u^0|; 0 \leq u \leq 1)$;*
3. *the Vervaat transform of Brownian motion $Z = (V_u; 0 \leq u \leq 1)$;*
4. *the first passage bridge through level $\lambda \neq 0$, $Z = (F_u^{\lambda,br}; 0 \leq u \leq 1)$;*
5. *the Vervaat transform of Brownian bridge with endpoint $\lambda \in \mathbb{R}$, Z=$(V_u^{\lambda}; 0 \leq u \leq 1)$.*

Note that in Theorem 1(4), (5), it suffices to consider the case of $\lambda < 0$ by time-reversal. As we will see later in Theorem 4, Theorem 1 is an immediate consequence of the fact that typical paths of these processes cannot be found in Brownian motion. The next theorem shows the possibility of embedding into Brownian motion some continuous-time processes whose distributions are singular with respect to Wiener measure.

Theorem 2 (Embeddings of Meander, Co-meander and 3-d Bessel Process) *For each of the following three processes $Z := (Z_u, 0 \leq u \leq 1)$ there is some random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as Z:*

1. *the meander $Z = (m_u; 0 \leq u \leq 1)$.*
2. *the co-meander $Z = (\tilde{m}_u; 0 \leq u \leq 1)$.*
3. *the three-dimensional Bessel process $Z = (R_u; 0 \leq u \leq 1)$.*

The problem of embedding Brownian bridge b^0 into Brownian motion is treated in a subsequent work of Pitman and Tang [73]. Since the proof relies heavily on Palm theory of stationary random measures, we prefer not to include it in the current work.

Theorem 3 ([73]) *There exists a random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as $(b_u^0; 0 \leq u \leq 1)$.*

In Question 1, we seek to embed a particular continuous-time process Z of unit length into a Brownian path. The distribution of X resides in the infinite-dimensional space $\mathcal{C}_0[0,1]$ of continuous paths $(w(t); 0 \leq t \leq 1)$ starting from $w(0) = 0$. So a closely related problem is whether a given subset of $\mathcal{C}_0[0,1]$ is hit by the path-valued moving-window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ indexed by $t \geq 0$. We formulate this problem as follows.

Question 2 Given a Borel measurable subset $S \subset \mathcal{C}_0[0,1]$, can we find a random time T such that $X_T := (B_{T+u} - B_T; 0 \leq u \leq 1) \in S$ with probability one?

Question 2 involves scanning for patterns in a continuous-time process. By the general theory of stochastic processes, assuming that the underlying Brownian motion B is defined on a complete probability space, $\{\exists T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S\}$ is measurable. See e.g. Dellacherie [20, T32, Chap. I], Meyer and Dellacherie [21, Sect. 44, Chap. III], and Bass [2, 3]. Assume that

$$\mathbb{P}(\exists T \geq 0 \text{ such that } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S) > 0.$$

Then there exists some fixed $M > 0$ and $p > 0$ such that

$$\mathbb{P}(\exists T : 0 \leq T \leq M \text{ and } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S) = p > 0.$$

We start the process afresh at $M + 1$, and then also

$$\mathbb{P}(\exists T : M + 1 \leq T \leq 2M + 1 \text{ and } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S) = p > 0.$$

By repeating the above procedure, we obtain a sequence of i.i.d. Bernoulli(p) random variables. Therefore, the probability that a given measurable set $S \subset \mathcal{C}_0[0, 1]$ is hit by the path-valued process generated by Brownian motion is either 0 or 1:

$$\mathbb{P}[\exists T \geq 0 \text{ such that } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S] = 0 \text{ or } 1. \tag{1}$$

Using various stochastic analysis tools, we are able to show that

Theorem 4 (Impossibility of Embedding of Excursion, Reflected Bridge, Vervaat Transform of Brownian Motion, First Passage Bridge and Vervaat Bridge Paths) *For each of the following five sets of paths S, almost surely, there is no random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S$:*

1. *the set of excursion paths, which first return to* 0 *at time* 1*,*

$$S = \mathcal{E} := \{w \in \mathcal{C}_0[0, 1]; w(t) > w(1) = 0 \text{ for } 0 < t < 1\};$$

2. *the set of reflected bridge paths,*

$$S = \mathcal{RBR}^0 := \{w \in \mathcal{C}_0[0, 1]; w(t) \geq w(1) = 0 \text{ for } 0 \leq t \leq 1\};$$

3. *the set of paths of Vervaat transform of Brownian motion with a floating negative endpoint,*

$$S = \mathcal{VB}^- := \{w \in \mathcal{C}_0[0, 1]; w(t) > w(1) \text{ for } 0 \leq t < 1 \text{ and } \inf\{t > 0; w(t) < 0\} > 0\};$$

4. *the set of first passage bridge paths at fixed level* $\lambda < 0$*,*

$$S = \mathcal{FP}^\lambda := \{w \in \mathcal{C}_0[0, 1]; w(t) > w(1) = \lambda \text{ for } 0 \leq t < 1\};$$

5. *the set of Vervaat bridge paths ending at fixed level* $\lambda < 0$,

$$S = \mathcal{VB}^{\lambda} := \{w \in \mathcal{FP}^{\lambda}; \inf\{t > 0; w(t) < 0\} > 0\} = \{w \in \mathcal{VB}^{-}; w(1) = \lambda\}.$$

Observe that for each $\lambda < 0$, $\mathcal{VB}^{\lambda}$ is a subset of $\mathcal{VB}^{-}$ and $\mathcal{FP}^{\lambda}$. Then Theorem 4(5) follows immediately from Theorem 4(3) or (4). As we will see in Sect. 3.1, Theorem 4(5) is also reminiscent of Theorem 4(1) in the proof.

It is obvious that for the following two sets of paths S, almost surely, there is a random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S$ almost surely:

- the set of positive paths,

$$S = \mathcal{M} := \{w \in \mathcal{C}_0[0, 1]; w(t) > 0 \text{ for } 0 < t \leq 1\};$$

- the set of bridge paths, which ends at $\lambda \in \mathbb{R}$,

$$S = \mathcal{BR}^{\lambda} := \{w \in \mathcal{C}_0[0, 1]; w(1) = \lambda\}.$$

The case of positive paths is easily treated by excursion theory, as discussed in Sect. 3.5. Bridge paths are obtained by simply taking $T := \inf\{t > 0; B_{t+1} = B_t + \lambda\}$, see Pitman and Tang [73] for related discussion. In both cases, $T+1$ is a stopping time relative to the Brownian filtration. For a general measurable $S \subset \mathcal{C}_0[0, 1]$, it is easily shown that if there is a random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S$ almost surely, then for each $\epsilon > 0$ this can be achieved by a random time T such that $T + 1 + \epsilon$ is a stopping time relative to the Brownian filtration.

In the current work, we restrict ourselves to continuous paths in linear Brownian motion. However, the problem is also worth considering in the multi-dimensional case, as discussed briefly in Sect. 4.

At first glance, neither Question 1 nor Question 2 seems to be tractable. To gain some intuition, we start by studying the analogous problem in the random walk setting. We deal with simple symmetric random walks $SW(n)$ of length n with increments ± 1 starting at 0. A typical question is how long it would take, in a random walk, to observe a pattern in a collection of patterns of length n satisfying some common properties. More precisely,

Question 3 Given for each $n \in \mathbb{N}$ a collection $\mathcal{A}^n$ of patterns of length $L(\mathcal{A}^n)$, what is the asymptotics of the expected waiting time $\mathbb{E}T(\mathcal{A}^n)$ until some element of $\mathcal{A}^n$ is observed in a random walk?

We are not aware of any previous study on pattern problems in which some natural definition of the collection of patterns is made for each $n \in \mathbb{N}$. Nevertheless, this question fits into the general theory of runs and patterns in a sequence of discrete trials. This theory dates back to work in 1940s by Wald and Wolfowitz [87] and Mood [66]. Since then, the subject has become important in various areas of science, including industrial engineering, biology, economics and statistics. In the 1960s, Feller [29] treated the problem probabilistically by identifying the occurrence of a single pattern as a renewal event. By the generating function method, the law of the occurrence times of a single pattern is entirely characterized. More advanced study, of the occurrence of patterns in a collection, developed in 1980s by two different methods. Guibas and Odlyzko [37], and Breen et al. [12] followed the steps of Feller [29] by studying the generating functions in pattern-overlapping regimes. An alternative approach was adopted by Li [55], and Gerber and Li [34] using martingale arguments. We also refer readers to the book of Fu and Lou [32] for the Markov chain embedding approach for multi-state trials.

Techniques from the theory of patterns in i.i.d. sequences provide general strategies to Question 3. Here we focus on some special cases where the asymptotics of the expected waiting time is computable. As we will see later, these asymptotics help us predict the existence or non-existence of some particular paths in Brownian motion. The following result answers Question 3 in some particular cases.

Theorem 5 *Let $T(\mathcal{A}^n)$ be the waiting time until some pattern in $\mathcal{A}^n$ appears in the simple random walk. Then*

1. *for the set of discrete positive excursions of length $2n$, whose first return to 0 occurs at time $2n$,*

$$\mathcal{E}^{2n} := \{w \in SW(2n); w(i) > 0 \text{ for } 1 \leq i \leq 2n-1 \text{ and } w(2n) = 0\},$$

we have

$$\mathbb{E}T(\mathcal{E}^{2n}) \sim 4\sqrt{\pi} n^{\frac{3}{2}}; \tag{2}$$

2. *for the set of positive walks of length $2n+1$,*

$$\mathcal{M}^{2n+1} := \{w \in SW(2n+1); w(i) > 0 \text{ for } 1 \leq i \leq 2n+1\},$$

we have

$$\mathbb{E}T(\mathcal{M}^{2n+1}) \sim 4n; \tag{3}$$

3. *for the set of discrete bridges of length n, which end at λ_n for some $\lambda \in \mathbb{R}$, where $\lambda_n := [\lambda\sqrt{n}]$ if $[\lambda\sqrt{n}]$ and n have the same parity, and $\lambda_n := [\lambda\sqrt{n}] + 1$ otherwise,*

$$\mathcal{BR}^{\lambda,n} := \{w \in SW(n); w(n) = \lambda_n\},$$

we have

$$c^{\lambda}_{\mathcal{BR}} n \leq \mathbb{E}T(\mathcal{BR}^{\lambda,n}) \leq C^{\lambda}_{\mathcal{BR}} n \quad \text{for some } c^{\lambda}_{\mathcal{BR}} \text{ and } C^{\lambda}_{\mathcal{BR}} > 0; \tag{4}$$

4. *for the set of negative first passage walks of length n, ending at* λ_n *with* $\lambda < 0$,

$$\mathcal{FP}^{\lambda,n} := \{w \in SW(n); w(i) > w(n) = \lambda_n \text{ for } 0 \le i \le n-1\},$$

we have

$$\sqrt{\frac{\pi}{2\lambda^2}} \exp\left(\frac{\lambda^2}{2}\right) n \le \mathbb{E}T(\mathcal{FP}^{\lambda,n}) \le \sqrt{\frac{4}{\lambda}} \exp\left(\frac{\lambda^2}{2}\right) n^{\frac{5}{4}}. \quad (5)$$

Now we explain how the asymptotics in Theorem 5 suggest answers to Question 1 and Question 2 in some cases. Formula (2) tells that it would take on average $n^{\frac{3}{2}} \gg n$ steps to observe an excursion in a simple random walk. In view of the usual scaling of random walks to converge to Brownian motion, the time scale appears to be too large. Thus we should not expect to find the excursion paths $\mathcal{E}$ in Brownian motion. However, in (3) and (4), the typical waiting time to observe a positive walk or a bridge has the same order n involved in the time scaling for convergence in distribution to Brownian motion. So we can anticipate to observe the positive paths $\mathcal{M}$ and the bridge paths $\mathcal{BR}^{\lambda}$ in Brownian motion. Finally in (5), there is an exponent gap in evaluating the expected waiting time for first passage walks ending at $\lambda_n \sim [\lambda\sqrt{n}]$ with $\lambda < 0$. In this case, we do not know whether it would take asymptotically n steps or much longer to first observe such patterns. This prevents us from predicting the existence of the first passage bridge paths $\mathcal{FP}^{\lambda}$ in Brownian motion.

The scaling arguments used in the last paragraph are quite intuitive but not rigorous since we are not aware of any theory which would justify the existence or non-existence of continuous paths in Brownian motion by taking limits from the discrete setting.

Organization of the Paper The rest of the paper is organized as follows.

- Section 2 treats the asymptotic behavior of the expected waiting time for discrete patterns. There Theorem 5 is proved.
- Section 3 is devoted to the analysis of continuous paths/processes in Brownian motion. Proofs of Theorems 2 and 4 are provided.
- Section 4 discusses the potential theory of continuous paths in Brownian motion.

A selection of open problems is presented in Sects. 2.5 and 4.

2 Expected Waiting Time for Discrete Patterns

Consider the expected waiting time for some collection of patterns

$$\mathcal{A}^n \in \{\mathcal{E}^{2n}, \mathcal{M}^{2n+1}, \mathcal{BR}^{\lambda,n}, \mathcal{FP}^{\lambda,n}\}$$

as defined in the introduction, except that we now encode a simple walk with m steps by its sequence of increments, with each increment a ± 1. We call such an increment sequence a *pattern* of length m. For each of these collections $\mathcal{A}^n$, all patterns in the collection have a common length, say $L(\mathcal{A}^n)$. We are interested in the asymptotic behavior of $\mathbb{E}T(\mathcal{A}^n)$ as $L(\mathcal{A}^n) \to \infty$.

We start by recalling the general strategy to compute the expected waiting time for discrete patterns in a simple random walk. From now on, let $\mathcal{A}^n := \{A_1^n, \cdots, A_{\#\mathcal{A}^n}^n\}$, where A_i^n is a sequence of signs ± 1 for $1 \le i \le \#\mathcal{A}^n$. That is,

$$A_i^n := A_{i1}^n \cdots A_{iL(\mathcal{A}^n)}^n, \quad \text{where } A_{ik}^n = \pm 1 \text{ for } 1 \le k \le L(\mathcal{A}^n).$$

Let $T(A_i^n)$ be the waiting time until the end of the first occurrence of A_i^n, and let $T(\mathcal{A}^n)$ be the waiting time until the first of the patterns in $\mathcal{A}^n$ is observed. So $T(\mathcal{A}^n)$ is the minimum of the $T(A_i^n)$ over $1 \le i \le \#\mathcal{A}^n$.

Define the matching matrix $M(\mathcal{A}^n)$, which accounts for the overlapping phenomenon among patterns within the collection $\mathcal{A}^n$. The coefficients are given by

$$M(\mathcal{A}^n)_{ij} := \sum_{l=0}^{L(\mathcal{A}^n)-1} \frac{\epsilon_l(A_i^n, A_j^n)}{2^l} \quad \text{for } 1 \le i, j \le \#\mathcal{A}^n, \tag{6}$$

where $\epsilon_l(A_i^n, A_j^n)$ is defined for $A_i^n = A_{i1}^n \cdots A_{iL(\mathcal{A}^n)}^n$ and $A_j^n = A_{j1}^n \cdots A_{jL(\mathcal{A}^n)}^n$ as

$$\epsilon_l(A_i^n, A_j^n) := \begin{cases} 1 & \text{if } A_{i1}^n = A_{j1+l}^n, \cdots, A_{iL(\mathcal{A}^n)-l}^n = A_{jL(\mathcal{A}^n)}^n \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

for $0 \le l \le L(\mathcal{A}^n) - 1$. Note that in general for $i \neq j$, $M(\mathcal{A}^n)_{ij} \neq M(\mathcal{A}^n)_{ji}$ and hence the matching matrix $M(\mathcal{A}^n)$ is not necessarily symmetric. The following result, which can be read from Breen et al. [12] is the main tool to study the expected waiting time for the collection of patterns.

Theorem 6 ([12])

1. *The matching matrix $M(\mathcal{A}^n)$ is invertible and the expected waiting times for patterns in $\mathcal{A}^n := \{A_1^n, \cdots, A_{\#\mathcal{A}^n}^n\}$ are given by*

$$\left(\frac{1}{\mathbb{E}T(A_1^n)}, \cdots, \frac{1}{\mathbb{E}T(A_{\#\mathcal{A}^n}^n)} \right)^T = \frac{1}{2^n} M(\mathcal{A}^n)^{-1} (1, \cdots, 1)^T; \tag{8}$$

2. *The expected waiting time till one of the patterns in $\mathcal{A}^n$ is observed is given by*

$$\frac{1}{\mathbb{E}T(\mathcal{A}^n)} = \sum_{l=1}^{\#\mathcal{A}^n} \frac{1}{\mathbb{E}T(A_l^n)} = \frac{1}{2^n} (1, \cdots, 1) M(\mathcal{A}^n)^{-1} (1, \cdots, 1)^T. \tag{9}$$

In Sect. 2.1, we apply the previous theorem to obtain the expected waiting time for discrete excursions $\mathcal{E}^{2n}$, i.e. (1) of Theorem 5. The same problem for positive walks $\mathcal{M}^{2n+1}$, bridge paths $\mathcal{BR}^{0,2n}$ and first passage walks $\mathcal{FP}^{\lambda,n}$ through $\lambda_n \sim \lambda\sqrt{n}$, i.e. (2)–(4) of Theorem 5, is studied in Sects. 2.2–2.4. Finally, we discuss the problem of the exponent gap for some discrete patterns in Sect. 2.5.

2.1 Expected Waiting Time for Discrete Excursions

For $n \in \mathbb{N}$, the number of discrete excursions of length $2n$ is equal to the $n-1$th Catalan number, see e.g. Stanley [82, Exercise 6.19 (i)]. That is,

$$\#\mathcal{E}^{2n} = \frac{1}{n}\binom{2n-2}{n-1} \sim \frac{1}{4\sqrt{\pi}} 2^{2n} n^{-\frac{3}{2}}. \tag{10}$$

Note that discrete excursions never overlap since the starting point and the endpoint are the only two minima. We have then $\epsilon(E_i^n, E_j^n) = \delta_{ij}$ for $1 \le i, j \le \#\mathcal{E}^{2n}$ by (7). Thus, the matching matrix defined as in (6) for discrete excursions $\mathcal{E}^{2n}$ has the simple form

$$M(\mathcal{E}^{2n}) = I_{\#\mathcal{E}^{2n}} \quad (\#\mathcal{E}^{2n} \times \#\mathcal{E}^{2n} \text{ identity matrix}).$$

According to Theorem 6,

$$\forall 1 \le i \le \#\mathcal{E}^{2n},\ \mathbb{E}T(E_i^n) = 2^{2n} \quad \text{and} \quad \mathbb{E}T(\mathcal{E}^n) = \frac{2^{2n}}{\#\mathcal{E}^{2n}} \sim 4\sqrt{\pi} n^{\frac{3}{2}}, \tag{11}$$

where $\#\mathcal{E}^{2n}$ is given as in (10). This is (2). □

2.2 Expected Waiting Time for Positive Walks

Let $n \in \mathbb{N}$. It is well-known that the number of non-negative walks of length $2n+1$ is $\binom{2n}{n}$, see e.g. Larbarbe and Marckert [52] and Leeuwen [84] for modern proofs. Thus the number of positive walks of length $2n+1$ is given by

$$\#\mathcal{M}^{2n+1} = \binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 2^{2n} n^{-\frac{1}{2}}. \tag{12}$$

Note that a positive walk of length $2n+1$ is uniquely determined by

- its first $2n$ steps, which is a positive walk of length $2n$;
- its last step, which can be either $+1$ or -1.

As a consequence,

$$\#\mathcal{M}^{2n} = \frac{1}{2}\#\mathcal{M}^{2n+1} \sim \frac{1}{\sqrt{\pi}} 2^{2n-1} n^{-\frac{1}{2}}. \tag{13}$$

Now consider the matching matrix $M(\mathcal{M}^{2n+1})$ defined as in (6) for positive walks $\mathcal{M}^{2n+1}$. $M(\mathcal{M}^{2n+1})$ is no longer diagonal since there are overlaps among positive walks. The following lemma presents the particular structure of this matrix.

Lemma 1 *$M(\mathcal{M}^{2n+1})$ is a multiple of some right stochastic matrix (whose row sums are equal to 1). The multiplicity is*

$$1 + \sum_{l=1}^{2n} \frac{k(\mathcal{M}^l)}{2^l} \sim \frac{2}{\sqrt{\pi}}\sqrt{n}. \tag{14}$$

Proof Let $1 \le i \le \#\mathcal{M}^{2n+1}$ and consider the sum of the ith row

$$\begin{aligned}\sum_{j=1}^{\#\mathcal{M}^{2n+1}} M(\mathcal{M}^{2n+1})_{ij} &:= \sum_{j=1}^{\#\mathcal{M}^{2n+1}} \sum_{l=0}^{2n} \frac{\epsilon_l(M_i^{2n+1}, M_j^{2n+1})}{2^l} \\ &= \sum_{l=0}^{2n} \frac{1}{2^l} \sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_l(M_i^{2n+1}, M_j^{2n+1}),\end{aligned} \tag{15}$$

where for $0 \le l \le 2n$ and $M_i^{2n+1}, M_j^{2n+1} \in \mathcal{M}^{2n+1}$, $\epsilon_l(M_i^{2n+1}, M_j^{2n+1})$ is defined as in (7). Note that $\epsilon_0(M_i^{2n+1}, M_j^{2n+1}) = 1$ if and only if $i = j$. Thus,

$$\sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_0(M_i^{2n+1}, M_j^{2n+1}) = 1. \tag{16}$$

In addition, for $1 \le l \le 2n$,

$$\begin{aligned}&\sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_l(M_i^{2n+1}, M_j^{2n+1}) \\ &\quad = \#\{M_j^{2n+1} \in \mathcal{M}^{2n+1}; M_{i1}^{2n+1} = M_{j1+l}^{2n+1}, \cdots, M_{i2n+1-l}^{2n+1} = M_{j2n+1}^{2n+1}\}.\end{aligned}$$

Note that given $M_{i1}^{2n+1} = M_{j1+l}^{2n+1}, \cdots, M_{i2n+1-l}^{2n+1} = M_{j2n+1}^{2n+1}$, which implies that $M_{j1+l}^{2n+1} \cdots M_{j2n+1}^{2n+1}$ is a positive walk of length $2n - l + 1$, we have

$$M_j^{2n+1} \in \mathcal{M}^{2n+1} \Longleftrightarrow M_{j1}^{2n+1} \cdots M_{jl}^{2n+1} \text{ is a positive walk of length } l.$$

Therefore, for $1 \le l \le 2n$,

$$\sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_l(M_i^{2n+1}, M_j^{2n+1}) = k(\mathcal{M}^l). \tag{17}$$

In view of (15), (16) and (17), we obtain for all $1 \le i \le \#\mathcal{M}^{2n+1}$, the sum of ith row of $M(\mathcal{M}^{2n+1})$ is given by (14). Furthermore, by (12) and (13), we know that $k(\mathcal{M}^l) \sim \frac{1}{\sqrt{2\pi}} 2^l l^{-\frac{1}{2}}$ as $l \to \infty$, which yields the asymptotics $\frac{2}{\sqrt{\pi}}\sqrt{n}$. □

Now by Theorem 6(1), $M(\mathcal{M}^{2n+1})$ is invertible and the inverse $M(\mathcal{M}^{2n+1})^{-1}$ is as well the multiple of some right stochastic matrix. The multiplicity is

$$\left(1 + \sum_{l=1}^{n-1} \frac{k(\mathcal{M}^l)}{2^l}\right)^{-1} \sim \frac{\sqrt{\pi}}{2\sqrt{n}}.$$

Then using (9), we obtain

$$\mathbb{E}T(\mathcal{M}^{2n+1}) = \frac{2^{2n+1}}{\left(1 + \sum_{l=1}^{n-1} \frac{k(\mathcal{M}^l)}{2^l}\right)^{-1} \#\mathcal{M}^{2n+1}} \sim 4n. \tag{18}$$

This is (3). □

2.3 Expected Waiting Time for Bridge Paths

In this part, we deal with the expected waiting time for the set of discrete bridges. In order to simplify the notations, we focus on the set of bridges of length $2n$ which end at $\lambda = 0$, that is $\mathcal{BR}^{0,2n}$. Note that the result in the general case for $\mathcal{BR}^{\lambda,n}$, where $\lambda \in \mathbb{R}$, can be derived in a similar way. Using Theorem 6, we prove a weaker version of (4): there exist $\tilde{c}^0_{\mathcal{BR}}$ and $C^0_{\mathcal{BR}} > 0$ such that

$$\tilde{c}^0_{\mathcal{BR}} n^{\frac{1}{2}} \le \mathbb{E}T(\mathcal{BR}^{0,n}) \le C^0_{\mathcal{BR}} n. \tag{19}$$

Compared to (4), there is an exponent gap in (19) and the lower bound is not optimal. Nevertheless, the lower bound of (4) follows a soft argument by scaling limit, Proposition 1. We defer the discussion to Sect. 2.5. It is standard that the number of discrete bridges of length $2n$ is

$$\#\mathcal{BR}^{0,2n} = \binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 2^{2n} n^{-\frac{1}{2}}. \tag{20}$$

Denote $\mathcal{BR}^{0,2n} := \{BR_1^{2n}, \cdots, BR_{\#\mathcal{BR}^{0,2n}}^{2n}\}$ and $M(\mathcal{BR}^{0,2n})$ the matching matrix of $\mathcal{BR}^{0,2n}$. We first establish the LHS estimate of (19). According to (8), we have

$$(1, \cdots, 1)M(\mathcal{BR}^{0,2n}) \left(\frac{1}{\mathbb{E}T(BR_1^{2n})}, \cdots, \frac{1}{\mathbb{E}T(BR_{\#\mathcal{BR}^{0,2n}}^{2n})} \right)^T = \frac{\#\mathcal{BR}^{0,2n}}{2^{2n}}. \tag{21}$$

Note that the matching matrix $\mathcal{M}(\mathcal{BR}^{0,2n})$ is non-negative with diagonal elements

$$M(\mathcal{BR}^{0,2n})_{ii} \geq \epsilon_0(BR_i^{2n}, BR_i^{2n}) = 1,$$

for $1 \leq i \leq \#\mathcal{BR}^{0,2n}$. As a direct consequence, the column sums of $M(\mathcal{BR}^{0,2n})$ is larger or equal to 1. Then by (9) and (21),

$$\mathbb{E}T(\mathcal{BR}^{0,2n}) \geq \frac{2^{2n}}{\#\mathcal{BR}^{0,2n}} \sim \sqrt{\pi n},$$

where $\#\mathcal{BR}^{0,2n}$ is defined as in (20). Take then $\tilde{c}_{\mathcal{BR}}^0 = \sqrt{\pi}$.

Now we establish the RHS estimate of (19). In view of (21), it suffices to work out an upper bound for the column sums of $M(\mathcal{BR}^{0,2n})$. Similarly as in (15), for $1 \leq j \leq \#\mathcal{BR}^{0,2n}$,

$$\sum_{i=1}^{\#\mathcal{BR}^{0,2n}} M(\mathcal{BR}^{0,2n})_{ij} = 1 + \sum_{l=1}^{2n-1} \frac{1}{2^l} \sum_{i=1}^{\#\mathcal{BR}^{0,2n}} \epsilon_l(BR_i^{2n}, BR_j^{2n}), \tag{22}$$

and

$$\begin{aligned} &\sum_{i=1}^{\#\mathcal{BR}^{0,2n}} \epsilon_l(BR_i^{2n}, BR_j^{2n}) \\ &\quad = \#\{BR_i^{2n} \in \mathcal{BR}^{0,2n}; BR_{i1}^{2n} = BR_{j1+l}^{2n}, \cdots, BR_{in-l}^{2n} = BR_{jn}^{2n}\}, \\ &\quad = \#\{\text{discrete bridges of length } l \text{ which end at } \sum_{k=1}^{n-l} BR_{jk}^{2n}\} \\ &\quad = \binom{l}{\frac{l+\sum_{k=1}^{n-l} BR_{jk}^{2n}}{2}} \leq \binom{l}{[\frac{l}{2}]}, \end{aligned} \tag{23}$$

where the last inequality is due to the fact that $\binom{l}{k} \leq \binom{l}{[l]/2}$ for $0 \leq k \leq l$. By (22) and (23), the column sums of $M(\mathcal{BR}^{0,2n})$ are bounded from above by

$$1 + \sum_{l=0}^{2n-1} \frac{1}{2^l} \binom{l}{[\frac{l}{2}]} \sim \frac{4}{\sqrt{\pi}} n^{\frac{1}{2}}.$$

Again by (9) and (21),

$$\mathbb{E}T(\mathcal{BR}^{0,2n}) \leq 2^{2n} \frac{4n^{\frac{1}{2}}/\sqrt{\pi}}{\#\mathcal{BR}^{0,2n}} \sim 4n.$$

Hence we take $C^0_{\mathcal{BR}} = 4$. □

2.4 Expected Waiting Time for First Passage Walks

We consider the expected waiting time for first passage walks through $\lambda_n \sim \lambda\sqrt{n}$ for $\lambda < 0$. Following Feller [29, Theorem 2, Chap. III.7], the number of patterns in $\mathcal{FP}^{\lambda,n}$ is

$$\#\mathcal{FP}^{\lambda,n} = \frac{\lambda_n}{n}\binom{n}{\frac{n+\lambda_n}{2}} \sim \lambda \exp\left(-\frac{\lambda^2}{2}\right)\sqrt{\frac{2}{\pi}}2^n n^{-1}. \tag{24}$$

For $\mathcal{FP}^{\lambda,n} := \{FP^n_1, \cdots, FP^n_{\#\mathcal{FP}^{\lambda,n}}\}$ and $M(\mathcal{FP}^{\lambda,n})$ the matching matrix for $\mathcal{FP}^{\lambda,n}$, we have, by (8), that

$$(1,\cdots,1)M(\mathcal{FP}^{\lambda,n})\left(\frac{1}{\mathbb{E}T(FP^n_1)},\cdots,\frac{1}{\mathbb{E}T(FP^n_{\#\mathcal{FP}^{\lambda,n}})}\right)^T = \frac{\#\mathcal{FP}^{\lambda,n}}{2^n}. \tag{25}$$

The LHS bound of (5) can be derived in a similar way as in Sect. 2.3.

$$\mathbb{E}T(\mathcal{FP}^{\lambda,n}) \geq \frac{2^n}{\#\mathcal{FP}^{\lambda,n}} \sim \sqrt{\frac{\pi}{2\lambda^2}}\exp\left(\frac{\lambda^2}{2}\right)n,$$

where $\#\mathcal{FP}^{\lambda,n}$ is defined as in (24). We get the lower bound of (5).

For the upper bound of (5), we aim to obtain an upper bound for the column sums of $M(\mathcal{FP}^{\lambda,n})$. Note that for $1 \leq j \leq k^{\lambda}_{\mathcal{FP}^n}$,

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij} = 1 + \sum_{l=1}^{n-1}\frac{1}{2^l}\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} \epsilon_l(FP_i, FP_j) \tag{26}$$

and

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} \epsilon_l(FP^n_i, FP^n_j)$$
$$= \#\{FP^n_i \in \mathcal{FP}^{\lambda,n}; FP^n_{i1} = FP^n_{j1+l}, \cdots, FP^n_{in-l} = FP^n_{jn}\}.$$

Observe that $\{FP_i^n \in \mathcal{FP}^{\lambda,n}; FP_{i1}^n = FP_{j1+l}^n, \cdots, FP_{in-l}^n = FP_{jn}^n\} \neq \emptyset$ if and only if $\sum_{k=1}^{l} FP_{jk}^n < 0$ (otherwise $\sum_{k=1}^{n-l} FP_{ik}^n = \sum_{k=1+l}^{n} FP_{jk}^n = \lambda_n - \sum_{k=1}^{l} FP_{jk}^n < \lambda_n$, which implies $FP_i^n \notin \mathcal{FP}^{\lambda,n}$). Then given $FP_{i1}^n = FP_{j1+l}^n, \cdots, FP_{in-l}^n = FP_{jn}^n$ and $\sum_{k=1}^{l} FP_{jk}^n < 0$,

$$FP_i^n \in \mathcal{FP}^{\lambda,n}$$

$$\Longleftrightarrow FP_{in-l+1}^n \cdots FP_{in}^n \text{ is a first passage walk of length } l \text{ through } \sum_{k=1}^{l} FP_{jk}^n < 0.$$

Therefore, for $1 \leq l \leq n-1$ and $1 \leq j \leq k_{\mathcal{FP}^n}^{\lambda}$,

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} \epsilon_l(FP_i^n, FP_j^n) = 1_{\sum_{k=1}^{l} FP_{jk}^n < 0} \frac{|\sum_{k=1}^{l} FP_{jk}^n|}{l} \binom{l}{\frac{l+\sum_{k=1}^{l} FP_{jk}^n}{2}}. \tag{27}$$

From the above discussion, it is easy to see for $1 \leq j \leq \#\mathcal{FP}^{\lambda,n}$,

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij} \leq \sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij^*},$$

where $FP_{j^*}^n$ is defined as follows: $FP_{j^*k}^n = -1$ if $1 \leq k \leq \lambda_n - 1$; $\lambda_n - 1 < k \leq n-1$ and $k - \lambda_n$ is odd; $k = n$. Otherwise $FP_{j^*k}^n = 1$.

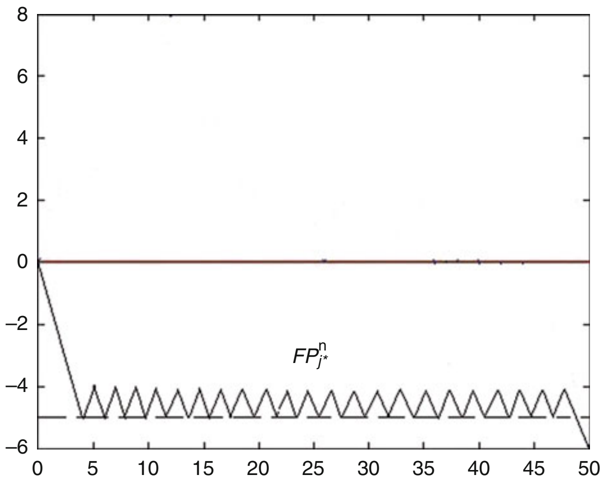

Fig. 1 Extreme patterns $FP_{j^*}^n$

The rest of this part is devoted to estimating $\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij^*}$. By (26) and (27),

$$\begin{aligned}
&\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij^*} \\
&= \sum_{l=0}^{|\lambda_n|-1} \frac{1}{2^l} + \sum_{\substack{l=\lambda_n \\ l-|\lambda_n| \text{ odd}}}^{n-1} \frac{|\lambda_n|-1}{l \cdot 2^l} \binom{l}{\frac{l-|\lambda_n|+1}{2}} + \sum_{\substack{l=\lambda_n \\ l-|\lambda_n| \text{ even}}}^{n-1} \frac{|\lambda_n|-2}{l \cdot 2^l} \binom{l}{\frac{l-|\lambda_n|+2}{2}} \\
&\leq 2 + |\lambda_n| \sum_{l=|\lambda_n|}^{n-1} \frac{1}{2^l l} \binom{l}{[\frac{l}{2}]} \sim \sqrt{\frac{8\lambda}{\pi}} n^{\frac{1}{4}}.
\end{aligned}$$

Thus, the column sums of $M(\mathcal{FP}^{\lambda,n})$ are bounded from above by $\sqrt{\frac{8\lambda}{\pi}} n^{\frac{1}{4}}$. By (9) and (25),

$$\mathbb{E}T(\mathcal{FP}^{\lambda,n}) \leq \frac{2^n \sqrt{8\lambda/\pi} n^{\frac{1}{4}}}{\#\mathcal{FP}^{\lambda,n}} \sim \sqrt{\frac{4}{\lambda}} \exp\left(\frac{\lambda^2}{2}\right) n^{\frac{5}{4}}.$$

This is the upper bound of (5). □

2.5 Exponent Gaps for $\mathcal{BR}^{\lambda,n}$ and $\mathcal{FP}^{\lambda,n}$

It can be inferred from (19) (resp. (5)) that the expected waiting time for $\mathcal{BR}^{\lambda,n}$ where $\lambda \in \mathbb{R}$ (resp. $\mathcal{FP}^{\lambda,n}$ where $\lambda < 0$) is bounded from below by order $n^{\frac{1}{2}}$ (resp. n) and from above by order n (resp. $n^{\frac{5}{4}}$). The exponent gap in the estimates of first passage walks $\mathcal{FP}^{\lambda,n}$ is frustrating, since we do not know whether the waiting time is exactly of order n, or is of order $\gg n$. This prevents the prediction of the existence of first passage bridge patterns $\mathcal{FP}^{\lambda}$ in Brownian motion.

From (9), we see that the most precise way to compute $\mathbb{E}T(\mathcal{BR}^{\lambda,n})$ and $\mathbb{E}T(\mathcal{FP}^{\lambda,n})$ consists in evaluating the sum of all entries in the inverse matching matrices $M(\mathcal{BR}^{\lambda,n})^{-1}$ and $M(\mathcal{FP}^{\lambda,n})^{-1}$. But the task is difficult since the structures of $M(\mathcal{BR}^{\lambda,n})$ and $M(\mathcal{FP}^{\lambda,n})$ are more complex than the structures of $M(\mathcal{E}^{2n})$ and $M(\mathcal{M}^{2n+1})$. We do not understand well the exact form of the inverse matrices $M(\mathcal{BR}^{\lambda,n})^{-1}$ and $M(\mathcal{FP}^{\lambda,n})^{-1}$.

The technique used in Sects. 2.3 and 2.4 is to bound the column sums of the matching matrix $M(\mathcal{BR}^{\lambda,n})$ (resp. $M(\mathcal{FP}^{\lambda,n})$). More precisely, we have proved that

$$\mathcal{O}(1) \leq \text{ column sums of } M(\mathcal{BR}^{\lambda,n}) \leq \mathcal{O}(n^{\frac{1}{2}}) \quad \text{for each fixed } \lambda \in \mathbb{R}; \tag{28}$$

$$\mathcal{O}(1) \leq \text{ column sums of } M(\mathcal{FP}^{\lambda,n}) \leq \mathcal{O}(n^{\frac{1}{4}}) \quad \text{for each fixed } \lambda < 0. \tag{29}$$

For the bridge pattern $\mathcal{BR}^{0,2n}$, the LHS bound of (28) is obtained by any excursion path of length $2n$, while the RHS bound of (28) is achieved by the sawtooth path with consecutive ± 1 increments. In the first passage pattern $\mathcal{FP}^{\lambda,n}$ where $\lambda < 0$, the LHS bound of (29) is achieved by some excursion-like path, which starts with an excursion and goes linearly to $\lambda\sqrt{n} < 0$ at the end. The RHS bound of (29) is given by the extreme pattern defined in Sect. 2.4, see Fig. 1. However, the above estimations are not accurate, since there are only few columns in $\mathcal{BR}^{\lambda,n}$ which sum up either to $\mathcal{O}(1)$ or to $\mathcal{O}(n^{\frac{1}{2}})$, and few columns of $\mathcal{FP}^{\lambda,n}$ which sum up either to $\mathcal{O}(1)$ or to $\mathcal{O}(n^{\frac{1}{4}})$.

Open Problem 1

1. Determine the exact asymptotics for $\mathbb{E}T(\mathcal{BR}^{\lambda,n})$ where $\lambda \in \mathbb{R}$, as $n \to \infty$.
2. Determine the exact asymptotics for $\mathbb{E}T(\mathcal{FP}^{\lambda,n})$ where $\lambda < 0$, as $n \to \infty$.

As we prove below, for $\lambda \in \mathbb{R}$, $\mathbb{E}T(\mathcal{BR}^{\lambda,n}) \asymp n$ by a scaling limit argument. Nevertheless, to obtain this result only by discrete analysis would be of independent interest. Table 1 provides the simulations of the expected waiting time $\mathbb{E}T(\mathcal{FP}^{-1,n})$ for some large n.

Table 1 Estimation of ζ by $\log \frac{\mathbb{E}T(\mathcal{FP}^{n_2}_{-1})}{\mathbb{E}T(\mathcal{FP}^{n_1}_{-1})} / \log(\frac{n_2}{n_1})$, where n_2 is the next to n_1 in the table

n	100	200	500	1000	2000	5000	10,000
$\mathbb{E}T(\mathcal{FP}^{n}_{-1})$	179.805	358.249	893.041	1800.002	3682.022	8549.390	12231.412
Estimated ζ		0.9945	0.9968	1.0112	1.0375	1.0205	1.0335

The result suggests that $\mathbb{E}T(\mathcal{FP}^{-1,n})$ be linear, but possibly with some log-correction. Yuval Peres made the following conjecture:

Conjecture 1 (Peres (personal communications)) For $\lambda < 0$, there exist $c^{\lambda}_{\mathcal{FP}}$ and $C^{\lambda}_{\mathcal{FP}} > 0$ such that

$$c^{\lambda}_{\mathcal{FP}} n \ln n \le \mathbb{E}T(\mathcal{FP}^{\lambda,n}) \le C^{\lambda}_{\mathcal{FP}} n \ln n. \tag{30}$$

This is consistent with Theorem 4(4), that we cannot find a first passage bridge with fixed negative endpoint in Brownian motion.

Now let us focus on the lower bound (4) of expected waiting time for bridge pattern $\mathcal{BR}^0$. For $n \in 2\mathbb{N}$, we run a simple random walk $(RW_k)_{k\in\mathbb{N}}$ until the first level bridge of length n appears. That is, we consider

$$(RW_{F_n+k} - RW_{F_n})_{0\le k\le n}, \quad \text{where } F_n := \inf\{k \ge 0; RW_{k+n} = RW_k\}. \tag{31}$$

For simplicity, let RW_k for non-integer k be defined by the usual linear interpolation of a simple random walk. For background on the weak convergence in $\mathcal{C}[0,1]$, we refer readers to Billingsley [11, Chap. 2].

Proposition 1 *The process*

$$\left(\frac{RW_{F_n+nu} - RW_{F_n}}{\sqrt{n}}; 0 \le u \le 1\right)$$

converges weakly in $\mathcal{C}[0,1]$ *to the bridge-like process*

$$(B_{F+u} - B_F; 0 \le u \le 1), \quad \textit{where } F := \inf\{t > 0; B_{t+1} - B_t = 0\}. \tag{32}$$

The process $(S_t := B_{t+1} - B_t; t \ge 0)$ is a stationary Gaussian process, first studied by Slepian [81] and Shepp [79]. The following result, which can be found in Pitman and Tang [73, Lemma 2.3], is needed for the proof of Proposition 1.

Lemma 2 ([71, 78]) *For each fixed* $t \ge 0$*, the distribution of* $(S_u; t \le u \le t+1)$ *is mutually absolutely continuous with respect to the distribution of*

$$(\tilde{B}_u := \sqrt{2}(\xi + B_u); t \le u \le t+1), \tag{33}$$

where $\xi \sim \mathcal{N}(0,1)$*. In particular, the distribution of the Slepian zero set restricted to* $[t, t+1]$*, i.e.* $\{u \in [t, t+1]; S_u = 0\}$ *is mutually absolutely continuous with respect to that of* $\{u \in [t, t+1]; \xi + B_u = 0\}$*, the zero set of Brownian motion starting at* $\xi \sim \mathcal{N}(0,1)$.

Proof of Proposition 1 Let $\mathbb{P}^{\mathbf{W}}$ be Wiener measure on $\mathcal{C}[0,\infty)$. Let $\mathbb{P}^{\mathbf{S}}$ (resp. $\mathbb{P}^{\tilde{\mathbf{W}}}$) be the distribution of the Slepian process S (resp. the distribution of $\tilde{B}$ defined as in (33)). We claim that

$$F := \inf\{t \ge 0; w_{t+1} = w_t\},$$

is a functional of the coordinate process $w := \{w_t; t \ge 0\} \in \mathcal{C}[0,\infty)$ that is continuous $\mathbb{P}^{\mathbf{W}}$ a.s. Note that the distribution of $(x_t := w_{t+1} - w_t; t \ge 0)$ under $\mathbb{P}^{\mathbf{W}}$ is the same as that of $(w_t; t \ge 0)$ under $\mathbb{P}^{\mathbf{S}}$. In addition, $x \in \mathcal{C}[0,\infty)$ is a functional of $w \in \mathcal{C}[0,\infty)$ that is continuous $\mathbb{P}^{\mathbf{W}}$ a.s. By composition, it is equivalent to show that

$$F' := \inf\{t \ge 0; w_t = 0\},$$

is a functional of $w \in \mathcal{C}[0,\infty)$ that is continuous $\mathbb{P}^{\mathbf{S}}$ a.s. Consider the set

$$\mathcal{Z} := \{w \in \mathcal{C}[0,\infty); F' \text{ is not continuous at } w\} = \cup_{p \in \mathbb{Q}} \mathcal{Z}_p,$$

where $\mathcal{Z}_p := \{w \in \mathcal{C}[0,\infty); F' \in [p, p+1]$ and F' is not continuous at $w\}$. It is obvious that $\mathbb{P}^{\tilde{\mathbf{W}}}(\mathcal{Z}) = 0$ and thus $\mathbb{P}^{\tilde{\mathbf{W}}}(\mathcal{Z}_p) = 0$ for all $p \geq 0$. By Lemma 2, $\mathbb{P}^{\mathbf{S}}$ is locally absolutely continuous relative to $\mathbb{P}^{\tilde{\mathbf{W}}}$, which implies that $\mathbb{P}^{\mathbf{S}}(\mathcal{Z}_p) = 0$ for all $p \geq 0$. As a countable union of null events, $\mathbb{P}^{\mathbf{S}}(\mathcal{Z}) = 0$, and the claim is proved. Thus, the mapping

$$\Xi_F : \mathcal{C}[0,\infty) \ni (w_t; t \geq 0) \longrightarrow (w_{F+u} - w_F; 0 \leq u \leq 1) \in \mathcal{C}[0,1]$$

is continuous $\mathbb{P}^{\mathbf{W}}$ a.s. According to Donsker's theorem [23], see e.g. Billingsley [11, Sect. 10] or Kallenberg [40, Chap. 16], the linearly interpolated simple random walks

$$\left(\frac{RW_{[nt]}}{\sqrt{n}}; t \geq 0\right) \text{ converges weakly in } \mathcal{C}[0,1] \text{ to } (B_t; t \geq 0).$$

So by the continuous mapping theorem, see e.g. Billingsley [11, Theorem 5.1],

$$\Xi_F \circ \left(\frac{RW_{[nt]}}{\sqrt{n}}; t \geq 0\right) \text{ converges weakly to } \Xi_F \circ (B_t; t \geq 0). \quad \square$$

Note that $T(\mathcal{BR}^{0,n}) = F_n + n$. Following the above analysis, we know that $T(\mathcal{BR}^{0,n})/n$ converges weakly to $F+1$, where $T(\mathcal{BR}^{0,n})$ is the waiting time until an element of $\mathcal{BR}^{0,n}$ occurs in a simple random walk and F is the random time defined as in (32). As a consequence,

$$\liminf_{n\to\infty} \mathbb{E}\frac{T(\mathcal{BR}^{0,n})}{n} \geq \mathbb{E}F + 1, \quad \text{since } \mathbb{E}F < \infty.$$

In particular, $\mathbb{E}F \leq C^0_{\mathcal{BR}} - 1 = 3$ as in Sect. 2.3. We refer readers to Pitman and Tang [73] for further discussion on first level bridges and the structure of the Slepian zero set.

3 Continuous Paths in Brownian Motion

This section is devoted to the proof of Theorems 2 and 4 regarding continuous paths and the distribution of continuous-time processes embedded in Brownian motion. In Sect. 3.1, we show that there is no normalized excursion in a Brownian path, i.e. Theorem 4(1). A slight modification of the proof allows us to exclude the existence of the Vervaat bridges with negative endpoint, i.e. Theorem 4(5). Furthermore, we prove in Sect. 3.2 that there is even no reflected bridge in Brownian motion, i.e. Theorem 4(2). In Sects. 3.3 and 3.4, we show that neither the Vervaat transform of Brownian motion nor first passage bridges with negative endpoint can be found in Brownian motion, i.e. Theorem 4(3), (4). We make use of the potential theory of *additive Lévy processes*, which is recalled in Sect. 3.3. Finally in Sect. 3.5, we provide a proof for the existence of Brownian meander, co-meander and three-dimensional Bessel process in Brownian motion, i.e. Theorem 2, using the filling scheme.

3.1 No Normalized Excursion in a Brownian Path

In this part, we provide two proofs for Theorem 4(1), though similar, from different viewpoints. The first proof is based on a fluctuation version of *Williams' path decomposition* of Brownian motion, originally due to Williams [88], and later extended in various ways by Millar [61, 62], and Greenwood and Pitman [36]. We also refer readers to Pitman and Winkel [72] for a combinatorial explanation and various applications.

Theorem 7 ([36, 88]) *Let $(B_t; t \geq 0)$ be standard Brownian motion and ξ be exponentially distributed with rate $\frac{1}{2}\vartheta^2$, independent of $(B_t; t \geq 0)$. Define $M := \text{argmin}_{[0,\xi]} B_t$, $H := -B_M$ and $R := B_\xi + H$. Then H and R are independent exponential variables, each with the same rate ϑ. Furthermore, conditionally given H and R, the path $(B_t; 0 \leq t \leq \xi)$ is decomposed into two independent pieces:*

- *$(B_t; 0 \leq t \leq M)$ is Brownian motion with drift $-\vartheta < 0$ running until it first hits the level $-H < 0$;*
- *$(B_{\xi-t} - B_\xi; 0 \leq t \leq \xi - M)$ is Brownian motion with drift $-\vartheta < 0$ running until it first hits the level $-R < 0$.*

Now we introduce the notion of first passage process, which will be used in the proof of Theorem 4(1). Given a càdlàg process $(Z_t; t \geq 0)$ starting at 0, we define the first passage process $(\tau_{-x}; x \geq 0)$ associated to X to be the first time that the level $-x < 0$ is hit:

$$\tau_{-x} := \inf\{t \geq 0; Z_t < -x\} \quad \text{for } x > 0.$$

When Z is Brownian motion, the distribution of the first passage process is well-known:

Lemma 3

1. *Let $\mathbf{W}$ be Wiener measure on $\mathcal{C}[0,\infty)$. Then the first passage process $(\tau_{-x}; x \geq 0)$ under $\mathbf{W}$ is a stable$(\frac{1}{2})$ subordinator, with*

$$\mathbb{E}^{\mathbf{W}}[\exp(-\alpha\tau_{-x})] = \exp(-x\sqrt{2\alpha}) \quad \textit{for } \alpha > 0.$$

2. *For $\vartheta \in \mathbb{R}$, let $\mathbf{W}^\vartheta$ be the distribution on $\mathcal{C}[0,\infty)$ of Brownian motion with drift ϑ.*

Then for each fixed $L > 0$, on the event $\tau_{-L} < \infty$, the distribution of the first passage process $(\tau_{-x}; 0 \leq x \leq L)$ under $\mathbf{W}^{\vartheta}$ is absolutely continuous with respect to that under $\mathbf{W}$, with density $D_L^{\vartheta} := \exp(-\vartheta L - \frac{\vartheta^2}{2}\tau_{-L})$.

Proof The part (1) of the lemma is a well known result of Lévy, see e.g. Bertoin et al. [8, Lemma 4]. The part (2) is a direct consequence of Girsanov's theorem, see e.g. Revuz and Yor [74, Chap. VIII] for background. □

Proof of Theorem 4(1) Suppose by contradiction that $\mathbb{P}(T < \infty) > 0$, where T is a random time at which some excursion appears. Take ξ exponentially distributed with rate $\frac{1}{2}$, independent of $(B_t; t \geq 0)$. We have then

$$\mathbb{P}(T < \xi < T + 1) > 0. \tag{34}$$

Now $(T, T+1)$ is inside the excursion of Brownian motion above its past-minimum process, which straddles ξ. See Fig. 2. Define

- $(\tau_{-x}; x \geq 0)$ to be the first passage process of $(B_{\xi+t} - B_{\xi}; t \geq 0)$.

By the strong Markov property of Brownian motion, $(B_{\xi+t} - B_{\xi}; t \geq 0)$ is still Brownian motion. Thus, $(\tau_{-x}; x \geq 0)$ is a stable$(\frac{1}{2})$ subordinator by part (1) of Lemma 3. Also, define

- $(\sigma_{-x}; x \geq 0)$ to be the first passage process derived from the process $(B_{\xi-t} - B_{\xi}; 0 \leq t \leq \xi - M)$ followed by an independent Brownian motion with drift -1 running forever.

According to Theorem 7, $(B_{\xi-t} - B_{\xi}; 0 \leq t \leq \xi - M)$ is Brownian motion with drift -1 running until it first hits the level $-R < 0$. Then $(\sigma_{-x}; x \geq 0)$ is the first

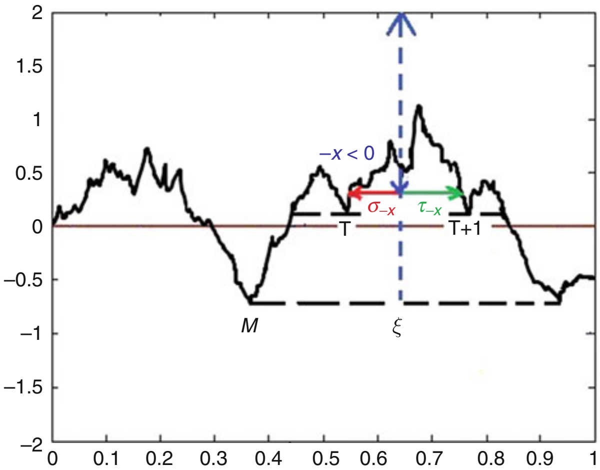

Fig. 2 No excursion of length 1 in a Brownian path

passage process of Brownian motion with drift −1, whose distribution is absolutely continuous on any compact interval $[0, L]$, with respect to that of $(\tau_{-x}; 0 \le x \le L)$ by part (2) of Lemma 3.

Thus, the distribution of $(\sigma_{-x} + \tau_{-x}; 0 \le x \le L)$ is absolutely continuous relative to that of $(\tau_{-2x}; 0 \le x \le L)$. It is well known that a real stable($\frac{1}{2}$) process does not hit points, see e.g. Bertoin [5, Theorem 16, Chap. II.5]. As a consequence,

$$\mathbb{P}(\sigma_{-x} + \tau_{-x} = 1 \text{ for some } x \ge 0) = 0,$$

which contradicts (34). □

Proof of Theorem 4(5) Impossibility of embedding the Vervaat bridge paths $\mathcal{VB}^\lambda$ with endpoint $\lambda < 0$. We borrow the notations from the preceding proof. Observe that, for fixed $\lambda < 0$,

$$\mathbb{P}(\sigma_{-x} + \tau_{-x+\lambda} = 1 \text{ for some } x \ge 0) = 0.$$

The rest of the proof is just a duplication of the preceding one. □

We give yet another proof of Theorem 4(1), which relies on Itô's excursion theory, combined with Bertoin's self-similar fragmentation theory. For general background on fragmentation processes, we refer to the monograph of Bertoin [7]. The next result, regarding a normalized Brownian excursion, follows Bertoin [6, Corollary 2].

Theorem 8 ([6]) *Let $e := (e_u; 0 \le u \le 1)$ be normalized Brownian excursion and $F^e := (F^e_t; t \ge 0)$ be the associated interval fragmentation defined as $F^e_t := \{u \in (0,1); e_u > t\}$. Introduce*

- $\lambda := (\lambda_t; t \ge 0)$ *the length of the interval component of F^e that contains U, independent of the excursion and uniformly distributed;*
- $\xi := \{\xi_t; t \ge 0\}$ *a subordinator, the Laplace exponent of which is given by*

$$\Phi^{ex}(q) := q\sqrt{\frac{8}{\pi}} \int_0^1 t^{q-\frac{1}{2}}(1-t)^{-\frac{1}{2}} = q\sqrt{\frac{8}{\pi}} B(q+\frac{1}{2}, \frac{1}{2}); \tag{35}$$

Then $(\lambda_t; t \ge 0)$ has the same law as $(\exp(-\xi_{\rho_t}); t \ge 0)$, where

$$\rho_t := \inf\left\{u \ge 0; \int_0^u \exp\left(-\frac{1}{2}\xi_r\right) dr > t\right\}. \tag{36}$$

Alternative Proof of Theorem 4(1) Consider the reflected process $(B_t - \underline{B}_t; t \ge 0)$, where $\underline{B}_t := \inf_{0 \le u \le t} B_u$ is the past-minimum process of the Brownian motion. For $\mathbf{e}$ the first excursion of $B - \underline{B}$ that contains some excursion pattern $\mathcal{E}$ of length 1, let $\Lambda_\mathbf{e}$ be the length of such excursion, and $\mathbf{e}^*$ be the normalized Brownian excursion. Following Itô's excursion theory, see e.g. Revuz and Yor [74, Chap. XII], $\Lambda_\mathbf{e}$ is independent of the distribution of the normalized excursion $\mathbf{e}^*$.

As a consequence, the fragmentation associated to $\mathbf{e}^*$ produces an interval of length $\frac{1}{\Lambda_{\mathbf{e}}}$. Now choose U uniformly distributed on $[0, 1]$ and independent of the Brownian motion. According to Theorem 8, there exists a subordinator ξ characterized as in (35) and a time-change ρ defined as in (36) such that $(\lambda_t; t \geq 0)$, the process of the length of the interval fragmentation which contains U, has the same distribution as $(\exp(-\xi_{\rho_t}); t \geq 0)$. Note that $(\lambda_t; t \geq 0)$ depends only on the normalized excursion $\mathbf{e}^*$ and U, so $(\lambda_t; t \geq 0)$ is independent of $\Lambda_{\mathbf{e}}$. It is a well known result of Kesten [41] that a subordinator without drift does not hit points. Therefore,

$$\mathbb{P}\left(\lambda_t = \frac{1}{\Lambda_{\mathbf{e}}} \text{ for some } t \geq 0\right) = 0,$$

which yields the desired result. □

3.2 No Reflected Bridge in a Brownian Path

This part is devoted to proving Theorem 4(2). The main difference between Theorem 4(1) and (2) is that the strict inequality $B_{T+u} > B_T$ for all $u \in (0, 1)$ is relaxed by the permission of equalities $B_{T+u} = B_T$ for some $u \in (0, 1)$. Thus, there are paths in $\mathcal{C}[0, 1]$ which may contain reflected bridge paths but not excursion paths. Nevertheless, such paths form a null set for Wiener measure. Below is a slightly stronger version of this result.

Lemma 4 *Almost surely, there are no random times $S < T$ such that $B_T = B_S$, $B_u \geq B_S$ for $u \in (S, T)$ and $B_v = B_w = B_S$ for some $S < v < w < T$.*

Proof Consider the following two sets

$$\mathcal{T} := \{\text{there exist } S \text{ and } T \text{ which satisfy the conditions in the lemma}\}$$

and

$$\mathcal{U} := \bigcup_{s,t \in \mathbb{Q}} \{B \text{ attains its minimum for more than once on } [s, t]\}.$$

It is straightforward that $\mathcal{T} \subset \mathcal{U}$. In addition, it is well-known that almost surely Brownian motion has a unique minimum on any fixed interval $[s, t]$ for all $s, t \in \mathbb{R}$. As a countable union of null events, $\mathbb{P}(\mathcal{U}) = 0$ and thus $\mathbb{P}(\mathcal{T}) = 0$. □

Remark 1 The previous lemma has an interesting geometric interpretation in terms of Brownian trees, see e.g. Pitman [71, Sect. 7.4] for background. Along the lines of the second proof of Theorem 4(1) in Sect. 3.1, we only need to show that the situation in Lemma 4 cannot happen in a Brownian excursion either of an independent and diffuse length or of normalized unit length. But this is just another way to state that Brownian trees have only binary branch points, which follows readily from Aldous' stick-breaking construction of the continuum random trees, see e.g. Aldous [1, Sect. 4.3] and Le Gall [53].

According to Theorem 4(1) and Lemma 4, we see that almost surely, there are neither excursion paths of length 1 nor reflected bridge paths of any length with at least two intermediate returns in Brownian motion. To prove the desired result, it suffices to exclude the possibility of reflected bridge paths with exactly one reflection. This is done by the following lemma.

Lemma 5 *Assume that* $0 \le S < T < U$ *are random times such that* $B_S = B_T = B_U$ *and* $B_u > B_S$ *for* $u \in (S, T) \cup (T, U)$. *Then the distribution of* $U - S$ *is absolutely continuous with respect to the Lebesgue measure.*

Proof Suppose by contradiction that the distribution of $U - S$ is not absolutely continuous with respect to the Lebesgue measure. Then there exists $p, q \in \mathbb{Q}$ such that $U - S$ fails to have a density on the event $\{S < p < T < q < U\}$. In fact, if $U - S$ has a density on $\{S < p < T < q < U\}$ for all $p, q \in \mathbb{Q}$, Radon-Nikodym theorem guarantees that $U - S$ has a density on $\{S < T < U\} = \cup_{p,q\in\mathbb{Q}}\{S < p < T < q < U\}$. Note that on the event $\{S < p < T < q < U\}$, U is the first time after q such that the Brownian motion B attains $\inf_{u\in[p,q]} B_u$ and obviously has a density. Again by Radon-Nikodym theorem, the distribution of $U - S$ has a density on $\{S < p < T < q < U\}$, which leads to a contradiction. □

Remark 2 The previous lemma can also be inferred from a fine study on local minima of Brownian motion. Neveu and Pitman [68] studied the renewal structure of local extrema in a Brownian path, in terms of Palm measure, see e.g. Kallenberg [40, Chap. 11].

More precisely, denote

- $\mathcal{C}$ to be the space of continuous paths on $\mathbb{R}$, equipped with Wiener measure $\mathbf{W}$;
- E to be the space of excursions with lifetime ζ, equipped with Itô measure $\mathbf{n}$.

Then the Palm measure of all local minima is the image of $\frac{1}{2}(\mathbf{n} \times \mathbf{n} \times \mathbf{W})$ by the mapping $E \times E \times \mathcal{C} \ni (e, e', w) \to \tilde{w} \in \mathcal{C}$ given by

$$\tilde{w}_t = \begin{cases} w_{t+\zeta(e')} & \text{if } t \le -\zeta(e'), \\ e'_{-t} & \text{if } -\zeta(e') \le t \le 0, \\ e_t & \text{if } 0 \le t \le \zeta(e), \\ w_{t-\zeta(e)} & \text{if } t \ge \zeta(e). \end{cases}$$

See Fig. 3. Using the notations of Lemma 5, an in-between reflected position T corresponds to a Brownian local minimum. Then the above discussion implies that $U - S$ is the sum of two independent random variables with densities and hence is diffuse. See also Tsirelson [83] for the i.i.d. uniform sampling construction of Brownian local minima, which reveals the diffuse nature of $U - S$.

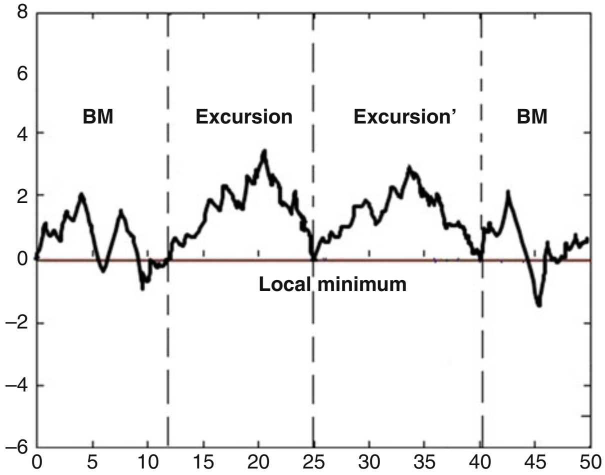

Fig. 3 Structure of local minima in Brownian motion

3.3 No Vervaat Transform of Brownian Motion in a Brownian Path

In the current section, we aim to prove Theorem 4(3). That is, there is no random time T such that

$$(B_{T+u} - B_T; 0 \leq u \leq 1) \in \mathcal{VB}^-.$$

A similar argument shows that there is no random time T such that

$$(B_{T+u} - B_T; 0 \leq u \leq 1) \in \mathcal{VB}^+,$$

where $\mathcal{VB}^+ := \{w \in \mathcal{C}[0,1]; w(t) > 0 \text{ for } 0 < t \leq 1 \text{ and } \sup\{t < 1; w(t) < w(1)\} < 1\}$. Observe that $(V_u; 0 \leq u \leq 1)$ is supported on $\mathcal{VB}^+ \cup \mathcal{VB}^-$. Thus, the Vervaat transform of Brownian motion cannot be embedded into Brownian motion.

In Sect. 3.1, we showed that for each fixed $\lambda < 0$, there is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in \mathcal{VB}^\lambda$. However, there is no obvious way to pass from the non-existence of the Vervaat bridges to that of the Vervaat transform of Brownian motion, due to an uncountable number of possible final levels.

To get around the problem, we make use of an additional tool–potential theory of additive Lévy processes, developed by Khoshnevisan et al. [43, 44, 47–49]. We now recall some results of this theory that we need in the proof of Theorem 4(3). For a more extensive overview of the theory, we refer readers to the survey of Khoshnevisan and Xiao [45].

Definition 1 An N-parameter, $\mathbb{R}^d$-valued additive Lévy process $(Z_{\mathbf{t}}; \mathbf{t} \in \mathbb{R}_+^N)$ with Lévy exponent $(\Psi^1, \dots, \Psi^N)$ is defined as

$$Z_{\mathbf{t}} := \sum_{i=1}^{N} Z_{t_i}^i \quad \text{for } \mathbf{t} = (t_1, \dots, t_N) \in \mathbb{R}_+^N, \tag{37}$$

where $(Z_{t_1}^1; t_1 \geq 0), \dots, (Z_{t_N}^N; t_N \geq 0)$ are N independent $\mathbb{R}^d$-valued Lévy processes with Lévy exponent $\Psi^1, \dots, \Psi^N$.

The following result regarding the range of additive Lévy processes is due to Khoshnevisan et al. [49, Theorem 1.5], [47, Theorem 1.1], and Yang [89, 90, Theorem 1.1].

Theorem 9 ([47, 49, 89]) *Let $(Z_{\mathbf{t}}; \mathbf{t} \in \mathbb{R}_+^N)$ be an additive Lévy process defined as in (37). Then*

$$\mathbb{E}[Leb(Z(\mathbb{R}_+^N))] > 0 \Longleftrightarrow \int_{\mathbb{R}^d} \prod_{i=1}^{N} Re\left(\frac{1}{1+\Psi^i(\zeta)}\right) d\zeta < \infty,$$

where $Leb(\cdot)$ is the Lebesgue measure on $\mathbb{R}^d$, and $Re(\cdot)$ is the real part of a complex number.

The next result, which is read from Khoshnevisan and Xiao [46, Lemma 4.1], makes a connection between the range of an additive Lévy process and the polarity of single points. See also Khoshnevisan and Xiao [45, Lemma 3.1].

Theorem 10 ([43, 46]) *Let $(Z_{\mathbf{t}}; \mathbf{t} \in \mathbb{R}_+^N)$ be an additive Lévy process defined as in (37). Assume that for each $\boldsymbol{t} \in \mathbb{R}_+^N$, the distribution of $Z_{\boldsymbol{t}}$ is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. Let $z \in \mathbb{R}^d \setminus \{0\}$, then*

$$\mathbb{P}(Z_{\boldsymbol{t}} = z \text{ for some } \boldsymbol{t} \in \mathbb{R}_+^N) > 0 \Longleftrightarrow \mathbb{P}(Leb(Z(\mathbb{R}_+^N)) > 0) > 0.$$

Note that $\mathbb{P}(Leb(Z(\mathbb{R}_+^N)) > 0) > 0$ is equivalent to $\mathbb{E}[Leb(Z(\mathbb{R}_+^N))] > 0$. Combining Theorems 9 and 10, we have:

Corollary 1 *Let $(Z_{\mathbf{t}}; \mathbf{t} \in \mathbb{R}_+^N)$ be an additive Lévy process defined as in (37). Assume that for each $\boldsymbol{t} \in \mathbb{R}_+^N$, the distribution of $Z_{\boldsymbol{t}}$ is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. Let $z \in \mathbb{R}^d \setminus \{0\}$, then*

$$\mathbb{P}(Z_{\boldsymbol{t}} = z \text{ for some } \boldsymbol{t} \in \mathbb{R}_+^N) > 0 \Longleftrightarrow \int_{\mathbb{R}^d} \prod_{i=1}^{N} Re\left(\frac{1}{1+\Psi^i(\zeta)}\right) d\zeta < \infty.$$

Proof of Theorem 4(3) We borrow the notations from the proof of Theorem 4(1) in Sect. 3.1. It suffices to show that

$$\mathbb{P}(\sigma_{-t_1} + \tau_{-t_2} = 1 \text{ for some } t_1, t_2 \geq 0) = 0, \tag{38}$$

where $(\sigma_{-t_1}; t_1 \geq 0)$ is the first passage process of Brownian motion with drift -1, and $(\tau_{-t_2}; t_2 \geq 0)$ is a stable($\frac{1}{2}$) subordinator independent of $(\sigma_{-t_1}; t_1 \geq 0)$. Let $Z_{\mathbf{t}} = Z^1_{t_1} + Z^2_{t_2} := \sigma_{-t_1} + \tau_{-t_2}$ for $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2_+$. By Definition 1, Z is a 2-parameter, real-valued additive Lévy process with Lévy exponent (Ψ^1, Ψ^2) given by

$$\Psi^1(\zeta) = \sqrt[4]{1 + 4\zeta^2} \exp\left[-i\frac{\arctan(2\zeta)}{2}\right] - 1 \quad \text{and} \quad \Psi^2(\zeta) = \sqrt{|\zeta|}(1 - i\, sgn \zeta)$$

for $\zeta \in \mathbb{R}$, which is derived from the formula in Cinlar [16, Chap. 7, Page 330] and Lemma 3(2). Hence,

$$Re\left(\frac{1}{1 + \Psi^1(\zeta)}\right) = \frac{1}{\sqrt[4]{1 + 4\zeta^2}} \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + 4\zeta^2}}\right)}$$

and

$$Re\left(\frac{1}{1 + \Psi^2(\zeta)}\right) = \frac{1 + \sqrt{|\zeta|}}{1 + 2\sqrt{|\zeta|} + 2|\zeta|}.$$

Clearly, $\Xi : \zeta \to Re\left(\frac{1}{1+\Psi^1(\zeta)}\right) Re\left(\frac{1}{1+\Psi^2(\zeta)}\right)$ is not integrable on $\mathbb{R}$ since $\Xi(\zeta) \sim \frac{1}{4|\zeta|}$ as $|\zeta| \to \infty$. In addition, for each $\mathbf{t} \in \mathbb{R}^2_+$, Z_t is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. Applying Corollary 1, we obtain (38). □

3.4 No First Passage Bridge in a Brownian Path

We prove Theorem 4(4), i.e. there is no first passage bridge in Brownian motion by a spacetime shift. The main difference between Vervaat bridges with fixed endpoint $\lambda < 0$ and first passage bridges ending at $\lambda < 0$ is that the former start with an excursion piece, while the latter return to the origin infinitely often on any small interval $[0, \epsilon]$, $\epsilon > 0$. Thus, the argument used in Sect. 3.1 to prove the non-existence of Vervaat bridges is not immediately applied in case of first passage bridges. Nevertheless, the potential theory of additive Lévy processes helps to circumvent the difficulty.

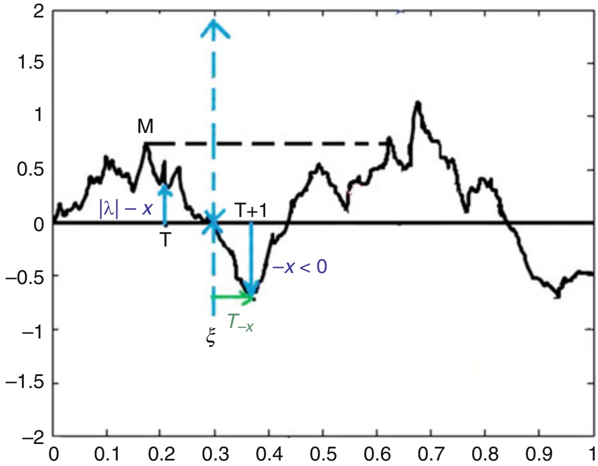

Fig. 4 No first passage bridge of length 1 in a Brownian path

Proof of Theorem 4(4) Suppose by contradiction that $\mathbb{P}(T < \infty) > 0$, where T is a random time that some first passage bridge through a fixed level appears. Take ξ exponentially distributed with rate $\frac{1}{2}$, independent of $(B_t; t \geq 0)$. We have then

$$\mathbb{P}(T < \xi < T + 1) > 0. \tag{39}$$

Now $(T, T+1)$ is inside the excursion of Brownian motion below its past-maximum process, which straddles ξ. See Fig. 4. Define

- $(\tau_{-x}; x \geq 0)$ to be the first passage process of $(B_{\xi+t} - B_\xi; t \geq 0)$.

By strong Markov property of Brownian motion, $(B_{\xi+t} - B_\xi; t \geq 0)$ is still Brownian motion. Thus, $(\tau_{-x}; x \geq 0)$ is a stable($\frac{1}{2}$) subordinator. Let $M := argmax_{[0,\xi]} B_t$. By a variant of Theorem 7, $(B_{\xi-t} - B_\xi; 0 \leq t \leq \xi - M)$ is Brownian motion with drift 1 running until it first hits the level $B_M - B_\xi > 0$, independent of $(\tau_{-x}; x \geq 0)$.

As a consequence, (39) implies that

$$\mathbb{P}(\tau_{-x} = l \text{ and } B^{\uparrow}_{1-l} = |\lambda| - x \text{ for some } (x, l) \in \mathbb{R}_+ \times [0, 1]) > 0, \tag{40}$$

where $(B^{\uparrow}_t; t \geq 0)$ is Brownian motion with drift 1, independent of $\frac{1}{2}$-stable subordinator $(\tau_{-x}; x \geq 0)$. By setting $t_1 := x$ and $t_2 := 1 - l$, we have:

$$\begin{aligned}
&\mathbb{P}(\tau_{-x} = l \text{ and } B^{\uparrow}_{1-l} = |\lambda| - x \text{ for some } (x, l) \in \mathbb{R}_+ \times [0, 1]) \\
&\quad = \mathbb{P}(\tau_{-t_1} + t_2 = 1 \text{ and } B^{\uparrow}_{t_2} + t_1 = |\lambda| \text{ for some } (t_1, t_2) \in \mathbb{R}_+ \times [0, 1]) \\
&\quad \leq \mathbb{P}[(\tau_{-t_1}, t_1) + (t_2, B^{\uparrow}_{t_2}) = (1, |\lambda|) \text{ for some } (t_1, t_2) \in \mathbb{R}^2_+]
\end{aligned} \tag{41}$$

Let $Z_{\mathbf{t}} = Z^1_{t_1} + Z^2_{t_2} := (\tau_{-t_1}, t_1) + (t_2, B^{\uparrow}_{t_2})$ for $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2_+$. By Definition 1, Z is a 2-parameter, $\mathbb{R}^2$-valued additive Lévy process with Lévy exponent (Ψ^1, Ψ^2) given by

$$\Psi^1(\zeta_1, \zeta_2) := \sqrt{|\zeta_1|} - i(\sqrt{|\zeta_1|} sgn \zeta_1 + \zeta_2) \quad \text{and} \quad \Psi^2(\zeta_1, \zeta_2) := \frac{\zeta_2^2}{2} - i(\zeta_1 + \zeta_2),$$

for $(\zeta_1, \zeta_2) \in \mathbb{R}^2$. Hence,

$$\begin{aligned} & Re\left(\frac{1}{1 + \Psi^1(\zeta_1, \zeta_2)}\right) Re\left(\frac{1}{1 + \Psi^2(\zeta_1, \zeta_2)}\right) \\ &= \frac{(1 + \sqrt{|\zeta_1|})\left(1 + \frac{\zeta_2^2}{2}\right)}{\left[(1 + \sqrt{|\zeta_1|})^2 + (\sqrt{|\zeta_1|} sgn \zeta_1 + \zeta_2)^2\right]\left[\left(1 + \frac{\zeta_2^2}{2}\right)^2 + (\zeta_1 + \zeta_2)^2\right]} := \Xi(\zeta_1, \zeta_2). \end{aligned}$$

Observe that $\zeta \to \Xi(\zeta_1, \zeta_2)$ is not integrable on $\mathbb{R}^2$, which is clear by passage to polar coordinates $(\zeta_1, \zeta_2) = (\rho \cos\theta, \sqrt{\rho} \sin\theta)$ for $\rho \geq 0$, $\theta \in [0, 2\pi)$. In addition, for each $\mathbf{t} \in \mathbb{R}^2_+$, $Z_{\mathbf{t}}$ is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$. Applying Corollary 1, we know that

$$\mathbb{P}(Z_{\mathbf{t}} = (1, |\lambda|) \text{ for some } \mathbf{t} \in \mathbb{R}^2_+) = 0.$$

Combining with (41), we obtain:

$$\mathbb{P}(\tau_{-x} = l \text{ and } B^{\uparrow}_{1-l} = |\lambda| - x \text{ for some } (x, l) \in \mathbb{R}_+ \times [0, 1]) = 0,$$

which contradicts (40). □

It is not hard to see that the above argument, together with those in Sect. 3.2 works for Bessel bridge of any dimension.

Corollary 2 (Impossibility of Embedding of Reflected Bridge Paths/Bessel Bridge) *For each fixed $\lambda > 0$, almost surely, there is no random time T such that*

$$\begin{aligned} &(B_{T+u} - B_T; 0 \leq u \leq 1) \in \mathcal{RBR}^{\lambda} \\ &\quad := \{w \in \mathcal{C}[0, 1]; w(t) \geq 0 \textit{ for } 0 \leq t \leq 1 \textit{ and } w(1) = \lambda\}. \end{aligned}$$

In particular, there is no random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as Bessel bridge ending at λ.

3.5 Meander, Co-meander and 3-d Bessel Process in a Brownian Path

We prove Theorem 2 in this section using Itô's excursion theory, combined with *Rost's filling scheme* [13, 75] solution to the Skorokhod embedding problem.

The existence of Brownian meander in a Brownian path is assured by the following well-known result, which can be read from Maisoneuve [58, Sect. 8], with explicit formulas due to Chung [15]. An alternative approach was provided by Greenwood and Pitman [35], and Pitman [70, Sects. 4 and 5]. See also Biane and Yor [9, Theorem 6.1], or Revuz and Yor [74, Exercise 4.18, Chap. XII].

Theorem 11 ([9, 35, 58]) *Let $(e^i)_{i \in \mathbb{N}}$ be the sequence of excursions, whose length exceeds 1, in the reflected process $(B_t - \underline{B}_t; t \geq 0)$, where $\underline{B}_t := \inf_{0 \leq u \leq t} B_u$ is the past-minimum process of the Brownian motion. Then $(e^i_u; 0 \leq u \leq 1)_{i \in \mathbb{N}}$ is a sequence of independent and identically distributed paths, each distributed as Brownian meander $(m_u; 0 \leq u \leq 1)$.*

Let us recall another basic result due to Imhof [38], which establishes the absolute continuity relation between Brownian meander and the three-dimensional Bessel process. Their relation with Brownian co-meander is studied in Yen and Yor [91, Chap. 7].

Theorem 12 ([38, 91]) *The distributions of Brownian meander $(m_u; 0 \leq u \leq 1)$, Brownian co-meander $(\tilde{m}_u; 0 \leq u \leq 1)$ and the three-dimensional Bessel process $(R_u; 0 \leq u \leq 1)$ are mutually absolutely continuous with respect to each other. For $F : \mathcal{C}[0, 1] \to \mathbb{R}^+$ a measurable function,*

1. $\mathbb{E}[F(m_u; 0 \leq u \leq 1)] = \mathbb{E}\left[\sqrt{\frac{\pi}{2}} \frac{1}{R_1} F(R_u; 0 \leq u \leq 1)\right]$;
2. $\mathbb{E}[F(\tilde{m}_u; 0 \leq u \leq 1)] = \mathbb{E}\left[\frac{1}{R_1^2} F(R_u; 0 \leq u \leq 1)\right]$.

According to Theorem 11, there exist $T_1, T_2, \cdots$ such that

$$m^i := (B_{T_i+u} - B_{T_i}; 0 \leq u \leq 1) \tag{42}$$

form a sequence of i.i.d. Brownian meanders. Since Brownian co-meander and the three-dimensional Bessel process are absolutely continuous relative to Brownian meander, it is natural to think of *von Neumann's acceptance-rejection algorithm* [86], see e.g. Rubinstein and Kroese [77, Sect. 2.3.4] for background and various applications. However, von Neumann's selection method requires that the Radon-Nikodym density between the underlying probability measures is essentially bounded, which is not satisfied in the cases suggested by Theorem 12. Nevertheless, we can apply the filling scheme of Chacon and Ornstein [13] and Rost [75].

We observe that sampling Brownian co-meander or the three-dimensional Bessel process from i.i.d. Brownian meanders $(m^i)_{i \in \mathbb{N}}$ fits into the general theory of Rost's filling scheme applied to the Skorokhod embedding problem. In the sequel, we follow the approach of Dellacherie and Meyer [22, Sects. 63–74, Chap. IX], which is based on the seminal work of Rost [75], to construct a stopping time N such that m^N achieves the distribution of $\tilde{m}$ or R. We need some notions from potential theory for the proof.

Definition 2

1. Given a Markov chain $X := (X_n)_{n\in\mathbb{N}}$, a function f is said to be excessive relative to X if

$$(f(X_n))_{n\in\mathbb{N}} \text{ is } \mathcal{F}_n - \text{supermartingale},$$

   where $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is the natural filtration of X.
2. Given two positive measures μ and λ, μ is said to be a balayage/sweeping of λ if

$$\mu(f) \leq \lambda(f) \quad \text{for all bounded excessive functions } f.$$

Proof of Theorem 2 Let μ^m (resp. μ^R) be the distribution of Brownian meander (resp. the three-dimensional Bessel process) on the space $(\mathcal{C}[0,1], \mathcal{F})$. By the filling scheme, the sequence of measures $(\mu_i^m, \mu_i^R)_{i\in\mathbb{N}}$ is defined recursively as

$$\mu_0^m := (\mu^m - \mu^R)^+ \quad \text{and} \quad \mu_0^R := (\mu^m - \mu^R)^-, \tag{43}$$

and for each $i \in \mathbb{N}$,

$$\mu_{i+1}^m := (\mu_i^m(1)\cdot\mu^m - \mu_i^R)^+ \quad \text{and} \quad \mu_{i+1}^R := (\mu_i^m(1)\cdot\mu^m - \mu_i^R)^-, \tag{44}$$

where $\mu_i^m(1)$ is the total mass of the measure μ_i^m. It is not hard to see that the bounded excessive functions of the i.i.d. meander sequence are constant μ^m a.s. Since μ^R is absolutely continuous with respect to μ^m, for each μ^m a.s. constant function c, $\mu^R(c) = \mu^m(c) = c$. Consequently, μ^R is a balayage/sweeping of μ^m by Definition 2. According to Theorem 69 of Dellacherie and Meyer [22],

$$\mu_\infty^R = 0, \quad \text{where } \mu_\infty^R :=\downarrow \lim_{i\to\infty} \mu_i^R.$$

Now let d_0 be the Radon-Nikodym density of μ_0^m relative to μ^m, and for $i > 0$, d_i be the Radon-Nikodym density of μ_i^m relative to $\mu_{i-1}^m(1)\cdot\mu^m$. We have

$$\begin{aligned}\mu^R &= (\mu^R - \mu_0^R) + (\mu_0^R - \mu_1^R) + \cdots \\ &= (\mu^m - \mu_0^m) + (\mu_0^m(1)\cdot\mu^m - \mu_1^m) + \cdots \\ &= (1-d_0)\mu^m + d_0\mu^m(1)\cdot(1-d_1)\mu^m + \cdots.\end{aligned} \tag{45}$$

Consider the stopping time N defined by

$$N := \inf \left\{ n \geq 0; -\sum_{i=0}^{n} \log d_i(m^i) > \xi \right\}, \tag{46}$$

where $(d_i)_{i\in\mathbb{N}}$ is the sequence of Radon-Nikodym densities defined as in the preceding paragraph, $(m^i)_{i\in\mathbb{N}}$ is the sequence of i.i.d. Brownian meanders defined as in (42), and ξ is exponentially distributed with rate 1, independent of $(m^i)_{i\in\mathbb{N}}$.

From the computation of (45), for all bounded measurable function f and all $k \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}[f(m^N); N = k] &= \mathbb{E}[f(m^k); -\sum_{i=0}^{k-1} \log d_i(m^i) \leq \xi < -\sum_{i=0}^{k} \log d_i(m^i)] \\
&= \mathbb{E}[d_0(m^0) \cdots d_{k-1}(m^{k-1}) f(m^k)(1 - d_k(m^k))] \\
&= (\mu_{k-1}^m(1) \cdot \mu^m - \mu_k^m) f \\
&= (\mu_{k-1}^R - \mu_k^R) f,
\end{aligned}$$

where $(\mu_i^m, \mu_i^R)_{i\in\mathbb{N}}$ are the filling measures defined as in (43) and (44). By summing over all k, we have

$$\mathbb{E}[f(m^N); N < \infty] = \mu^R f.$$

That is, m^N has the same distribution as R. As a summary,

$$(B_{T_N+u} - B_{T_N}; 0 \leq u \leq 1) \text{ has the same distribution as } (R_u; 0 \leq u \leq 1),$$

where $(T_i)_{i\in\mathbb{N}}$ are defined by (42) and N is the stopping time as in (46). Thus we achieve the distribution of the three-dimensional Bessel process in Brownian motion. The embedding of Brownian co-meander into Brownian motion is obtained in the same vein. □

Remark 3 Note that the stopping time N defined as in (46) has infinite mean, since

$$\mathbb{E}N = \sum_{i\in\mathbb{N}} \mu_i^m(1) = \infty.$$

The problem whether Brownian co-meander or the three-dimensional Bessel process can be embedded in finite expected time, remains open. More generally, Rost [76] was able to characterize all stopping distributions of a continuous-time Markov process, given its initial distribution. In our setting, let $(P_t)_{t\geq 0}$ be the semi-group of the moving window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$, and μ^W be its initial distribution, corresponding to Wiener measure on $\mathcal{C}[0, 1]$. Following Rost

[76], for any distribution μ on $\mathcal{C}[0,1]$, one can construct the continuous-time filling measures $(\mu_t, \mu_t^W)_{t \geq 0}$ and a suitable stopping time T such that

$$\mu - \mu_t + \mu_t^W = \mu^W P_{t \wedge T}.$$

Thus, the distribution μ is achieved if and only if $\mu_\infty = 0$, where $\mu_\infty :=\downarrow \lim_{t \to \infty} \mu_t$. In particular, Brownian motion with drift $(\vartheta t + B_t; 0 \leq t \leq 1)$ for a fixed ϑ, can be obtained for a suitable stopping time $T + 1$.

4 Potential Theory for Continuous-Time Patterns

In Question 2, we ask for any Borel measurable subset S of $\mathcal{C}_0[0,1]$ whether S is hit by the moving-window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$, at some random time T. Related studies of the moving window process appear in several contexts. Knight [50, 51] introduced the prediction processes, where the whole past of the underlying process is tracked to anticipate its future behavior. The relation between Knight's prediction processes and our problems is discussed briefly at the end of the section. Similar ideas are found in stochastic control theory, where certain path-dependent stochastic differential equations were investigated, see e.g. the monograph of Mohammed [63] and Chang et al. [14]. More recently, Dupire [25] worked out a functional version of Itô's calculus, in which the underlying process is path-valued and notions as time-derivative and space-derivative with respect to a path, are proposed. We refer readers to the thesis of Fournié [31] as well as Cont and Fournié [17–19] for further development.

Indeed, Question 2 is some issue of potential theory. In Benjamini et al. [4] a potential theory was developed for transient Markov chains on any countable state space E. They showed that the probability for a transient chain to ever visit a given subset $S \subset E$, is estimated by $Cap_M(S)$—the *Martin capacity* of the set S. See also Mörters and Peres [67, Sect. 8.3] for a detailed exposition. As pointed out by Steven Evans (personal communications), such a framework still works well for our discrete patterns. For $0 < \alpha < 1$, define the α-potential of the discrete patterns/strings of length n as

$$\begin{aligned} G^\alpha(\epsilon', \epsilon'') &:= \sum_{k=0}^{\infty} \alpha^k P^k(\epsilon', \epsilon'') \\ &= \sum_{k=0}^{n-1} \left(\frac{\alpha}{2}\right)^k 1\{\sigma_k(\epsilon') = \tau_k(\epsilon'')\} + \frac{1}{1-\alpha}\left(\frac{\alpha}{2}\right)^k, \end{aligned}$$

where $\epsilon', \epsilon'' \in \{-1,1\}^n$, and $P(\cdot,\cdot)$ is the transition kernel of discrete patterns/strings of length n in a simple random walk, and σ_k (resp. τ_k): $\{-1,1\}^n \to \{-1,1\}^{n-k}$ the restriction operator to the last $n-k$ strings (resp. to the first $n-k$ strings). The following result is a direct consequence of the first/second moment method, and we leave the detail to readers.

Proposition 2 (Evans (personal communications)) *Let T be an $\mathbb{N}$-valued random variable with $\mathbb{P}(T > n) = \alpha^n$, independent of the simple random walk. For $\mathcal{A}^n$ a collection of discrete patterns of length n, we have*

$$\frac{1}{2}\frac{2^{-n}}{1-\alpha}Cap_\alpha(\mathcal{A}^n) \leq \mathbb{P}(T(\mathcal{A}^n) < T) \leq \frac{2^{-n}}{1-\alpha}Cap_\alpha(\mathcal{A}^n),$$

where for $A \subset \{-1,1\}^n$,

$$Cap_\alpha(A) := \left[\inf\left\{\sum_{\epsilon',\epsilon''\in\{-1,1\}^n} G^\alpha(\epsilon',\epsilon'')g(\epsilon')g(\epsilon''); g \geq 0, g(A^c) = \{0\} \right.\right.$$
$$\left.\left. \text{and} \sum_{\epsilon\in\{0,1\}^n} g(\epsilon) = 1\right\}\right]^{-1}.$$

Now let us mention some previous work regarding the potential theory for path-valued Markov processes. There has been much interest in developing a potential theory for the Ornstein-Uhlenbeck process in the Wiener space $\mathcal{C}_0[0,\infty)$, defined as

$$Z_t := U(t,\cdot) \quad \text{for } t \geq 0,$$

where $U(t,\cdot) := e^{-t/2}W(e^t,\cdot)$ is the Ornstein-Uhlenbeck Brownian sheet. Note that the continuous-time process $(Z_t; t \geq 0)$ takes values in the Wiener space $\mathcal{C}_0[0,\infty)$ and starts at $Z_0 := W(1,\cdot)$ as standard Brownian motion. Following Williams [60], a Borel measurable set $S \subset \mathcal{C}_0[0,\infty)$ is said to be quasi-sure if $\mathbb{P}(\forall t \geq 0, Z_t \in S) = 1$, which is known to be equivalent to

$$Cap_{OU}(S^c) = 0, \tag{47}$$

where

$$Cap_{OU}(S^c) := \int_0^\infty e^{-t}\mathbb{P}(\exists T \in [0,t] \text{ such that } Z_T \in S^c)dt \tag{48}$$

is the *Fukushima-Malliavin capacity* of S^c, that is the probability that Z hits S^c before an independent exponential random time with parameter 1. Taking advantage of the well-known *Wiener-Itô decomposition* of the Ornstein-Uhlenbeck semigroup, Fukushima [33] provided an alternative construction of (47) via the Dirichlet form. The approach allows the strengthening of many Brownian almost sure properties to quasi-sure properties. See also the survey of Khoshnevisan [42] for recent development.

Note that the definition (48) can be extended to any (path-valued) Markov process. Within this framework, a related problem to Question 2 is

Question 4 Given a Borel measurable set $S_\infty \subset \mathcal{C}_0[0,\infty)$, is

$$\begin{aligned} Cap_{MW}(S_\infty) &:= \int_0^\infty e^{-t}\mathbb{P}[\exists T \in [0,t] \text{ such that } \Theta_T \circ B \in S_\infty]dt \\ &= 0 \text{ or } > 0? \end{aligned}$$

where $(\Theta_t)_{t\geq 0}$ is the family of spacetime shift operators defined as

$$\Theta_t \circ B := (B_{t+u} - B_t; u \geq 0) \quad \text{for all } t \geq 0. \tag{49}$$

It is not difficult to see that the set function Cap_{MW} is a Choquet capacity associated to the shifted process $(B_{t+u} - B_t; u \geq 0)$ for $t \geq 0$, or the moving-window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$. For a Borel measurable subset S of $\mathcal{C}_0[0,1]$, if $Cap_{MW}(S \otimes_1 \mathcal{C}_0[0,\infty)) = 0$, where

$$S \otimes_1 \mathcal{C}_0[0,\infty) := \{(w_t 1_{t<1} + (w_1 + w'_t)1_{t\geq 1})_{t\geq 0}; w \in S \text{ and } w' \in \mathcal{C}_0[0,\infty)\} \tag{50}$$

is the usual path-concatenation, then

$$\mathbb{P}(\exists T > 0 \text{ such that } X_T \in S] = 0,$$

i.e. almost surely the set S is not hit by the moving-window process X. Otherwise,

$$\mathbb{P}[\exists T \in [0,t] \text{ such that } X_T \in S] > 0 \quad \text{for some } t \geq 0,$$

and an elementary argument leads to $\mathbb{P}[\exists T \geq 0 \text{ such that } X_T \in S] = 1$.

As context for this question, we note that path-valued Markov processes have also been extensively investigated in the superprocess literature. In particular, Le Gall [54] characterized the polar sets for the Brownian snake, which relies on earlier work on the potential theory of symmetric Markov processes by Fitzsimmons and Getoor [30] among others.

There has been much progress in the development of potential theory for symmetric path-valued Markov processes. However, the shifted process, or the moving-window process, is not time-reversible and the transition kernel is more complicated than that of the Ornstein-Uhlenbeck process in Wiener space. So working with a non-symmetric Dirichlet form, see e.g. the monograph of Ma and Röckner [57], seems to be far from obvious.

Open Problem 2

1. Is there any relation between the two capacities Cap_X and Cap_{MW} on Wiener space?
2. Propose a non-symmetric Dirichlet form for the shifted process $(\Theta_t \circ B)_{t \geq 0}$, which permits to compute the capacities of the sets of paths $\mathcal{E}$, $\mathcal{M}$, $\mathcal{BR}^{\lambda}$ ...etc.

This problem seems substantial already for one-dimensional Brownian motion. But it could of course be posed also for higher dimensional Brownian motion, or a still more general Markov process. Following are some well-known examples of non-existing patterns in d-dimensional Brownian motion for $d \geq 2$.

- $d = 2$ (Evans [28]): There is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has a two-sided cone point with angle $\alpha < \pi$;
- $d = 3$ (Dvoretzky et al. [27]): There is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ contains a triple point;
- $d \geq 4$ (Kakutani [39], Dvoretzky et al. [26]): There is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ contains a double point.

We refer readers to the book of Mörters and Peres [67, Chaps. 9 and 10] for historical notes and further discussions on sample path properties of Brownian motion in all dimensions.

Finally, we make some connections between Knight's prediction processes and our problems. For background, readers are invited to Knight [50, 51] as well as the commentary of Meyer [59] on Knight's work. To avoid heavy measure theoretic discussion, we restrict ourselves to the classical Wiener space $(\mathcal{C}_0[0,\infty), \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^{\mathbf{W}})$, where $(\mathcal{F}_t)_{t \geq 0}$ is the augmented Brownian filtrations satisfying the usual hypothesis of right-continuity.

The prediction process is defined as, for all $t \geq 0$ and S_∞ a Borel measurable set of $\mathcal{C}_0[0,\infty)$,

$$Z_t^{\mathbf{W}}(S_\infty) := \mathbb{P}^{\mathbf{W}}[\Theta_t \circ B \in S_\infty | \mathcal{F}_t],$$

where $\Theta_t \circ B$ is the shifted path defined as in (49). Note that $(Z_t^{\mathbf{W}})_{t \geq 0}$ is a strong Markov process, which takes values in the space of probability measure on the Wiener space $(\mathcal{C}_0[0,\infty), \mathcal{F})$. In terms of the prediction process, Question 2 can be reformulated as

Question 5 Given a Borel measurable set $S \subset \mathcal{C}_0[0,1]$, can we find a random time T such that

$$\mathbb{E} Z_T^{\mathbf{W}}(S \otimes_1 \mathcal{C}_0[0,\infty)) = 1?$$

where $S \otimes_1 \mathcal{C}_0[0,\infty)$ is defined as in (50).

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