

THE CONVERGENCE RATE OF VANISHING VISCOSITY APPROXIMATIONS FOR MEAN FIELD GAMES

WENPIN TANG, YUMING PAUL ZHANG

ABSTRACT. Motivated by numerical challenges in first-order mean field games (MFGs) and the weak noise theory for the Kardar–Parisi–Zhang equation, we consider the problem of vanishing viscosity approximations for MFGs. We provide the first results on the convergence rate to the vanishing viscosity limit in mean field games, with a focus on the dimension dependence of the rate exponent. Two cases are studied: MFGs with a local coupling and those with a nonlocal, regularizing coupling. In the former case, we use a duality approach and our results suggest that there may be a phase transition in the dimension dependence of vanishing viscosity approximations in terms of the growth of the Hamiltonian and the local coupling. In the latter case, we rely on the regularity analysis of the solution, and derive a faster rate compared to MFGs with a local coupling. A list of open problems are presented.

Key words: Convergence rate, dimension dependence, duality, KPZ equation, local/nonlocal coupling, mean field games, vanishing viscosity approximations.

AMS 2010 Mathematics Subject Classification: 35K10, 35Q89, 49L25, 91A15

1. INTRODUCTION

Mean Field Games (MFGs) are a mathematical framework used to model and analyze strategic interactions in a large population, which was independently developed by Lasry and Lions [41, 42, 43], and Caines, Huang and Malhamé [36]. In MFGs, each individual makes decisions based on her own objective as well as the behavior of the entire population, represented by a mean field which describes the probability distribution of the collective state of the system. MFGs are widely used to model complex systems in economics [17, 38] and engineering [22, 35, 36].

The standard form of MFGs is given by the following system of partial differential equations (PDEs):

$$\begin{cases} -\partial_t u_\nu - \nu \Delta u_\nu + H(x, Du_\nu) = f(x, m_\nu), \\ \partial_t m_\nu - \nu \Delta m_\nu - \nabla \cdot (m_\nu D_p H(x, Du_\nu)) = 0, & \text{in } \Omega := (0, T) \times \mathbb{T}^d, \\ m_\nu(0, x) = \bar{m}(x), \quad u_\nu(T, x) = \bar{u}(x), \end{cases} \quad (1.1)$$

where $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ is the d -dimensional torus, $T > 0$ and $\nu \geq 0$. The Hamiltonian $H(x, p)$ is a convex function with respect to the second variable p . MFGs are used to describe Nash equilibria in differential games with a continuum of players, where $m_\nu(t, x)$ is the density of players at time t and at position x . The variable u_ν is the value of a typical player's optimal control problem, so it is a solution to some Hamilton–Jacobi equation. As u_ν is the optimal value (that the player can possibly achieve), the optimal strategy is $-D_p H(x, Du_\nu)$. If the

Date: March 25, 2023.

density of the population flows to the direction preferred by the optimal strategy, the game has a Nash equilibrium. This amounts to solving (1.1). When $\nu > 0$, the system is of second order. The system by sending ν to zero is called the *vanishing viscosity limit*, which is of first order.

In this paper, we study the convergence rate of second-order MFGs to the vanishing viscosity limit as $\nu \rightarrow 0^+$. A special focus is on the dependence of the rate on the dimension d , and hence on whether or under what circumstances it induces the curse of dimensionality or the lack thereof. For simplicity, we assume that the terminal data \bar{u} is independent of the density m_ν . We distinguish two cases for the coupling (or the running cost) f :

- When $f(x, m)$ depends on the pointwise value of the density $m(t, x)$, the coupling is referred to as *local*.
- When $f(x, m)$ depends on the entire distribution of $m(t, \cdot)$ and is uniformly smooth for all distributions, it is referred to as a *regularizing, nonlocal coupling*.

We assume that f is increasing in m in the local case, and satisfies the Lasry-Lions monotonicity condition in the nonlocal case, so that the uniqueness of solutions is guaranteed. The precise definitions and assumptions will be provided in Sections 2 and 6. It is also worth mentioning that the convergence rate of vanishing viscosity approximations of Hamilton-Jacobi equations was studied in [20, 23, 56, 57], and the optimal rate is $\nu^{\frac{1}{2}}$. The same problem was considered for hyperbolic systems in [2, 4, 5], and for Fokker-Planck equations with C^1 nonlocal drifts in [25, 26, 60].

Before delving into the problem, we digress a bit to explain the motivations to study the convergence of vanishing viscosity in MFGs.

- (1) In recent years, there has been a growing interest in modeling autonomous vehicles' control and their macroscopic traffic flow by first-order MFGs ($\nu = 0$) [33, 34, 39]. As pointed out in [33], numerical methods converge slowly, or even fail to converge for first-order MFGs. This is not surprising as iterative algorithms may be ill-posed due to irregular coefficients in the transport equation. On the other hand, it is known [6, 7, 8] that the policy iteration algorithm converges exponentially fast for second-order MFGs ($\nu > 0$). So a reasonable idea is to approximate first-order MFGs by second-order MFGs, and a quantitative rate of second-order MFGs to the vanishing viscosity limit provides the approximation error. Moreover, the convergence of the policy iteration algorithm is exponential in ν^{-1} , which yields a tradeoff between bias and algorithm efficiency.
- (2) The Kardar-Parisi-Zhang (KPZ) universality class describes the limiting behavior of a collection of random growth models, and the underlying continuum object is the KPZ equation [19, 55]. Due to its nonlinear nature, the KPZ equation is hardly accessible except for some initial conditions which lead to integrability. Recently, there has been a line of work on large deviations of the stochastic heat equation (SHE), and hence the $(1+1)$ -dimensional KPZ equation by the Cole-Hopf transform, under the *weak noise theory* [40, 48]. Rigorous treatments have been developed in [27, 45, 46, 59]. Under narrow wedge initial condition, the "most probable" KPZ path conditioned to be λ at time T is given by $h(t, x) = \log Z[\rho^m](t, x)$, where $Z[\rho]$ given $\rho = \rho(t, x)$ solves the PDE

$$\partial_t Z = \frac{1}{2} \partial_{xx} Z + \rho Z, \text{ for } (t, x) \in (0, T] \times \mathbb{R}, \quad Z(0, \cdot) = \delta_0,$$

and ρ^m solves the variational problem $\inf \{ \frac{1}{2} \|\rho\|_{L^2}^2 : Z[\rho](T, 0) = e^\lambda \}$. Of particular interest is the lower-tail limit as $\lambda := -\nu^{-1} \rightarrow -\infty$ (so $\nu \rightarrow 0^+$). Setting the scaling $Z_\nu[\rho](t, x) := Z[\nu^{-1}\rho(\cdot, \nu^{\frac{1}{2}}\cdot)](t, \nu^{-\frac{1}{2}}x)$, $\rho_\nu(t, x) := \nu\rho^m(t, \nu^{-\frac{1}{2}}x)$, and $h_\nu(t, x) := \nu^{-1} \log Z_\nu[\rho_\nu](t, x)$, [46, 59] showed that h_ν converges locally uniformly to a limiting shape h_* as $\nu \rightarrow 0^+$ by integrability of h_* . Curiously, (h_ν, ρ_ν) solves the PDEs:

$$\begin{cases} \partial_t h_\nu = \frac{\nu}{2} \partial_{xx} h_\nu + \frac{1}{2} (\partial_x h_\nu)^2 + \rho_\nu, \\ -\partial_t \rho_\nu = \frac{\nu}{2} \partial_{xx} \rho_\nu - \nabla \cdot (\rho_\nu \partial_x h_\nu), \end{cases} \quad (1.2)$$

with a suitable choice of the initial-terminal conditions (by a Riemann-Hilbert approach). By taking $H(x, p) = \frac{1}{2}p^2$ and $f(x, m) = m$ (a local coupling), and letting

$$h_\nu(t, x) := -u_{\nu/2}(T - t, x), \quad \rho_\nu(t, x) := -m_{\nu/2}(T - t, x),$$

the equations (1.1) specify to (1.2). Thus, our result on the convergence of second-order MFGs to the vanishing viscosity limit stipulates how the lower-tail limit of the most probable KPZ path in (1 + 1)-dimension is obtained, and gives a quantitative rate in the large deviation limit thereof. Of course, the Cole-Hopf transform from the SHE to the KPZ equation and the weak noise theory is only valid in dimension $d = 1$. There is no obvious theory for dimension $d \geq 2$ (see [11, 18, 47] for recent development), so it is not clear how large deviations of the KPZ equation in dimension $(d + 1)$ with $d \geq 2$ is connected to MFGs. Nevertheless, our results for MFGs hold for general dimensions.

Now we turn back to MFGs. The well-posedness of (1.1) has been thoroughly studied in the case of nonlocal, monotone, and regularizing couplings for both second-order MFGs ($\nu > 0$) and first-order MFGs ($\nu = 0$) [41, 42, 43], and for time-dependent MFGs [24]. When the function $f(x, m)$ depends locally on the value of m and $\nu > 0$, the well-posedness problem has been addressed by [28, 29] for classical solutions, and by [51, 52] for weak solutions. When $\nu = 0$, in general, one cannot expect the existence of classical solutions. [13] exploited the variational method, and showed that first-order MFGs can be viewed as an optimality condition for two convex problems. This approach can also be used for quadratic Hamiltonian MFGs in the whole domain [50]. Recently, allowing the terminal condition \bar{u} to depend on the density $m_\nu(T, \cdot)$, [49] obtained the weak solution as the limit of a sequence of classical solutions to strictly elliptic problems.

Here is an overview of our main results on the convergence rate of vanishing viscosity approximations for MFGs. For both local and nonlocal couplings, it is known that as $\nu \rightarrow 0^+$, the solutions (u_ν, m_ν) converge to (u, m) , where (u, m) are solutions to first-order MFGs [12, 15]. However, the convergence rate is not well understood. This paper provides the first quantitative rate of vanishing viscosity approximations for MFGs.

- (a) **Local coupling.** When the coupling f is local, we apply a duality approach which relies on the fact that equation (1.1) is the optimality condition for two convex optimization problems. This approach was used in [54], and can be traced back to [3]. To obtain a convergence rate of solutions as $\nu \rightarrow 0^+$, the main assumption is the coercivity of the Hamiltonian and the coupling, as specified in the conditions (H5-1) and (H5-2). Similar assumptions were made in [31, 54] to get a Sobolev estimate for m . With these assumptions, we prove that (u_ν, m_ν) converges in some Sobolev norm at a polynomial rate as $\nu \rightarrow 0^+$ (see Theorem 4.3).

To illustrate, take $H(x, p) = |p|^r$ and $f(x, m) = m^{q-1}$ for some $q, r > 1$ (while our results hold for more general Hamiltonian and local couplings). Let

$$\beta := \max \left\{ \frac{qr}{2qr - q - r}, 1 \right\} (d+1) - d \geq 1.$$

Our result shows that

$$\iint_{\Omega} (m_{\nu}^{q/2} - m^{q/2})^2 dxdt \lesssim \nu^{\frac{1}{1+\beta}}, \quad (1.3a)$$

$$\iint_{\Omega} \left| |Du_{\nu}|^{r/2-1} Du_{\nu} - |Du|^{r/2-1} Du \right|^2 m dxdt \lesssim \nu^{\frac{1}{1+\beta}}. \quad (1.3b)$$

In particular, we have $\iint_{\Omega} (m_{\nu} - m)^2 dxdt \lesssim \nu^{\frac{2}{q(1+\beta)}}$ for $q \geq 2$ (see Remark 4.2). Note that if $\frac{1}{q} + \frac{1}{r} \leq 1$, then $\beta = 1$ and the rate in (1.3b) is $\nu^{\frac{1}{2}}$ which is independent of the dimension d ; if $\frac{1}{q} + \frac{1}{r} > 1$, then the rate becomes $\nu^{\frac{c}{d}}$ for some $c > 0$ which decays slowly as the dimension d is large. So there is no curse of dimensionality if the growth of H and f is sufficiently large (i.e. $\frac{1}{q} + \frac{1}{r} \leq 1$), while the convergence may suffer from the curse of dimensionality otherwise. However, we do not know whether the rate $\nu^{\frac{1}{1+\beta}}$ is tight; if it is, or the rate is $\nu^{\kappa(d)}$ with any κ decreasing to 0 as $d \rightarrow \infty$ for $\frac{1}{q} + \frac{1}{r} > 1$, then it implies a phase transition in the dimension dependence of vanishing viscosity approximations for MFGs at $\frac{1}{q} + \frac{1}{r} = 1$.

When the Hamiltonian is quadratic ($q = 2$), we prove a stronger convergence result for u_{ν} :

$$\iint_{\Omega} |u_{\nu} - u|^2 m dxdt \lesssim \nu^{\frac{1}{2(1+\beta)}} \quad \text{if } r > \max \left\{ 2 + \frac{d}{d+1}, \frac{d^2}{2d+3} \right\}, \quad (1.4)$$

(see Theorem 5.3). This result, presented in Theorem 5.3, cannot be directly deduced from (1.3b), as the weighted Poincaré inequality may not be applicable. Instead, we use the higher regularity of m_{ν} (see [31]) and the equations. For $d = 1$, the condition in (1.4) requires $r > \frac{5}{2}$, which fails to cover the KPZ Hamiltonian with $r = 2$. To address this, we further need $d \leq 3$ to get uniform boundedness of u_{ν} in the KPZ setting with $q = r = 2$, as presented in Theorem 5.4.

We also mention that if the terminal data $\bar{u} = \bar{u}(x, m_{\nu}(T, x))$ depends on m_{ν} , the convergence rate of vanishing viscosity for MFGs remains open. In this case, we have no longer the optimality condition characterization.

- (b) **Nonlocal and regularizing coupling.** When the coupling is nonlocal and regularizing, we adopt a different approach. Instead of using the optimization structure, we rely on the duality of the two equations and some regularity properties of the solutions, both uniform and non-uniform with respect to ν . Our main tool is the uniform semi-concavity of the solution u_{ν} , which results from the superlinear growth of the Hamiltonian and classical Hamilton-Jacobi equations [10]. This property allows us to establish a uniform bound on the $W^{1,2}$ norm of $\nu^{\frac{1}{2}} m_{\nu}$ for all $\nu > 0$. With these findings, we first prove (1.3b) with $r = 2$ and $\beta = 1$, and

$$\iint_{\Omega} (f(x, m_{\nu}) - f(x, m))(m_{\nu} - m) dxdt \lesssim \nu^{\frac{1}{2}}, \quad (1.5)$$

(see Theorem 6.2). This method, while requiring less restrictive assumptions, leads to a faster convergence rate compared to those with a local coupling.

Assuming that (1.5) implies pointwise convergence of $f(x, m_\nu)$ to $f(x, m)$ as $\nu \rightarrow 0^+$, we obtain a pointwise convergence of u_ν with rate $\nu^{\frac{1}{4}}$ (see Theorem 7.2). While under a weaker condition (H4''), the estimate (1.5) implies that $f(x, m_\nu)$ converges to $f(x, m)$ in $L^1(\Omega)$ with a rate $\nu^{1/4}$. Then we use both the dual equation method (see [44]) and the viscosity solution method (see [20]) to prove that the HJ equation of u is stable under both L^1 -perturbation of coefficients and vanishing viscosity. Indeed, we obtain for all $t \leq T$,

$$\|u_\nu(t, \cdot) - u(t, \cdot)\|_{L^1(\Omega)} \lesssim \nu^{\frac{1}{4}},$$

(see Theorem 7.3).

We also mention that in the literature, \bar{m} is often assumed to be strictly positive, see [14, 15, 49, 53] for superlinear Hamiltonian, and [30] for linear Hamiltonian, or there are extra requirements on H and f [13, 32]. Relying on [14, 15], we show that the condition can be dropped without further restrictions (see Theorem 3.2).

The remainder of the paper is organized as follows. In Section 2, we provide background on MFGs with a local coupling. In Section 3, we prove the well-posedness of MFGs with nonnegative data. In Sections 4 and 5, we study the convergence rate of vanishing viscosity for MFGs with a local coupling. In Sections 6 and 7 we consider the convergence rate of vanishing viscosity for MFGs with a nonlocal and regularizing coupling. Finally, a list of open problems are presented in Section 8.

2. ASSUMPTIONS AND PRELIMINARIES FOR LOCAL COUPLING

We first discuss the assumptions for the case of local coupling f . The assumptions are made so that (1.1) is well-posedness for all $\nu \geq 0$, and most of them can be found in [14, 15]. We assume that there exists $C_0 \geq 1$ such that:

- (H1) (Conditions on the coupling) $f : \mathbb{T}^d \times [0, \infty) \rightarrow \mathbb{R}$ is continuous in both variables, strictly increasing with respect to the second variable, and there exists $q > 1$ such that

$$C_0^{-1}m^{q-1} - C_0 \leq f(x, m) \leq C_0m^{q-1} + C_0 \quad \text{for all } m \geq 0 \text{ and } x \in \mathbb{T}^d.$$

Moreover, we make the normalization condition:

$$f(x, 0) = 0 \quad \text{for all } x \in \mathbb{T}^d. \quad (2.1)$$

- (H2) (Conditions on the Hamiltonian) The Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous in both variables, strictly convex and differentiable in the second variable, with $D_p H$ continuous, and satisfying for some $r > 1$,

$$C_0^{-1}|p|^r - C_0 \leq H(x, p) \leq C_0|p|^r + C_0 \quad \text{for all } (x, p) \in \mathbb{T}^d \times \mathbb{R}^d.$$

- (H3) (Conditions on the initial and terminal data) $\bar{u} : \mathbb{T}^d \rightarrow \mathbb{R}$ is of class \mathcal{C}^2 , and $\bar{m} : \mathbb{T}^d \rightarrow \mathbb{R}$ is a \mathcal{C}^1 nonnegative density function.

Remark 2.1. *1. As discussed in [15], (2.1) is just a normalization condition, which can be assumed without loss of generality. In fact, if it does not hold, one can replace $f(x, m)$ and $H(x, p)$ by $f(x, m) - f(x, 0)$ and $H(x, p) - f(x, 0)$, respectively.*

2. Recall that the Fenchel conjugate $H^*(x, \cdot)$ of $H(x, \cdot)$ for each $x \in \mathbb{T}^d$ is defined as $H^*(x, \xi) := \sup_{p \in \mathbb{R}^d} (\langle \xi, p \rangle - H(x, p))$. Then H^* is continuous and satisfies for some $C_0 \geq 1$ (without loss of generality let us still use C_0) such that

$$C_0^{-1}|\xi|^{r'} - C_0 \leq H^*(x, \xi) \leq C_0|\xi|^{r'} + C_0 \quad \text{for all } (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d,$$

where $r' := \frac{r}{r-1}$ is the conjugate of r . Later we also write $q' := \frac{q}{q-1}$ as the conjugate of q .

3. Let F be defined as

$$F(x, m) := \int_0^m f(x, s) ds \quad \text{if } m \geq 0,$$

and $F(x, m) = +\infty$ if $m < 0$. Then (H2) yields for some $C_0 \geq 1$,

$$C_0^{-1}m^q - C_0 \leq F(x, m) \leq C_0m^q + C_0 \quad \text{for all } m \geq 0 \text{ and } x \in \mathbb{T}^d. \quad (2.2)$$

We define $F^*(x, \cdot)$ to be the Fenchel conjugate of $F(x, \cdot)$. Then $F^*(x, \alpha)$ is strictly convex in α , $F^*(x, \alpha) = 0$ for $\alpha \leq 0$, and for some $C_0 \geq 1$,

$$C_0^{-1}\alpha^{q'} - C_0 \leq F^*(x, \alpha) \leq C_0\alpha^{q'} + C_0 \quad \text{for all } \alpha \geq 0 \text{ and } x \in \mathbb{T}^d. \quad (2.3)$$

4. Unlike [14, 15], we do not need to assume $\bar{m} > 0$. It was used in [14, 15] to show the existence of a solution for the optimization problem (2.6).

5. One can consider equations with more general second order terms:

$$\begin{cases} -\partial_t u_\nu - \nu A_{ij} \partial_{ij} u_\nu + H(x, Du_\nu) = f(x, m_\nu), \\ \partial_t m_\nu - \nu \partial_{ij} (A_{ij} m_\nu) - \nabla \cdot (m_\nu D_p H(x, Du_\nu)) = 0, \\ m_\nu(0, x) = \bar{m}(x), \quad u_\nu(T, x) = \bar{u}(x) \end{cases}$$

where $A = A(x)$ is assumed to be a Lipschitz continuous map, taking values in the set of symmetric, uniformly positive definite matrices. But for this, one needs to further assume $r \geq q'$, see [15].

2.1. Optimization problems. We discuss two optimal control problems which are in duality, and we refer to [13, 15]. For any $\nu \geq 0$, the first problem is

$$\inf_{(m, w) \in \mathcal{K}_{1, \nu}} \mathcal{B}(m, w) \quad (2.4)$$

where

$$\mathcal{B}(m, w) := \iint_{\Omega} m H^* \left(x, -\frac{w}{m} \right) + F(x, m) dx dt + \int_{\mathbb{T}^d} \bar{u}(x) m(T, x) dx,$$

with $\Omega = (0, T) \times \mathbb{T}^d$, and

$$\mathcal{K}_{1, \nu} := \left\{ (m, w) \in L^1(\mathbb{T}^d) \times L^1(\mathbb{T}^d; \mathbb{R}^d) \mid \partial_t m - \nu \Delta m + \nabla \cdot w = 0, m(0) = \bar{m} \right\}$$

where the continuity equation holds in the sense of distributions. When $m = 0$, we use the usual convention:

$$m H^* \left(x, -\frac{w}{m} \right) := \begin{cases} +\infty, & \text{if } m = 0 \text{ and } w \neq 0, \\ 0, & \text{if } m = 0 \text{ and } w = 0. \end{cases}$$

Now we discuss the second minimization problem. Recall $q, r > 1$ from (H1)(H2) and q', r' are their conjugates, respectively. Set

$$\gamma := \frac{rq'(d+1)}{d-r(q'-1)} \text{ if } q' < 1 + \frac{d}{r} \quad \text{and} \quad \gamma := \infty \text{ if } q' > 1 + \frac{d}{r}, \quad (2.5)$$

and $\gamma > 0$ can be an arbitrarily large constant when $q' = 1 + \frac{d}{r}$. Then we let

$$\mathcal{A}(u, \alpha) := \iint_{\Omega} F^*(x, \alpha(t, x)) dx dt - \int_{\mathbb{T}^d} u(0, x) \bar{m}(x) dx,$$

and let $\mathcal{K}_{2,\nu}$ be the set of $(u, \alpha) \in L^\gamma(\Omega) \times L^{q'}(\Omega)$ such that $Du \in L^r(\Omega)$ and the following holds in the sense of distributions

$$-\partial_t u - \nu \Delta u + H(x, Du) \leq \alpha, \quad u(T, \cdot) \leq \bar{u}.$$

The precise meaning of the inequality is given in [15, Section 3]. The second optimization problem (also called the relaxed problem in [13, 15]) is

$$\inf_{(u, \alpha) \in \mathcal{K}_{2,\nu}} \mathcal{A}(u, \alpha). \quad (2.6)$$

It turns out that for each $\nu \geq 0$, the optimization problems (2.4) and (2.6) are in duality. The first equality below is proved in [13, 15] by the Fenchel-Rockafellar theorem. The second equality is due to the fact that one can always replace α by $\max\{\alpha, 0\}$ as $F^*(x, \alpha) = 0$ for $\alpha \leq 0$. Here we remark that $\bar{m} > 0$ is not needed in the proofs.

Theorem 2.1 ([15]). *For all $\nu \geq 0$,*

$$- \min_{(m, w) \in \mathcal{K}_{1,\nu}} \mathcal{B}(m, w) = \inf_{(u, \alpha) \in \mathcal{K}_{2,\nu}} \mathcal{A}(u, \alpha) = \inf_{(u, \alpha) \in \mathcal{K}_{2,\nu}, \alpha \geq 0 \text{ a.e.}} \mathcal{A}(u, \alpha).$$

Moreover, the minimum of the first term is achieved by a unique pair in $(m, w) \in \mathcal{K}_{1,\nu}$ satisfying $(m, w) \in L^q(\Omega) \times L^{\frac{r'q}{r'+q-1}}(\Omega)$.

The last statement that the minimum is achieved by a unique pair in $\mathcal{K}_{1,\nu}$ is because the set $\mathcal{K}_{1,\nu}$ is convex, and the functions $F(x, \cdot)$ and $H^*(x, \cdot)$ are strictly convex for each x .

3. WELL-POSEDNESS FOR NON-NEGATIVE DATA

In this section, we show well-posedness of (1.1) without assuming \bar{m} to be strictly positive. First we recall the notion of weak solutions in [15]. Let $\nu \geq 0$ and γ be from (2.5).

Definition 3.1. *We say that $(u_\nu, m_\nu) \in L^\gamma(\Omega) \times L^q(\Omega)$ is a weak solution to (1.1) if*

(i) *the following integrability conditions hold:*

$$Du_\nu \in L^r, \quad m_\nu H^*(\cdot, D_p H(\cdot, Du_\nu)) \in L^1 \quad \text{and} \quad m_\nu D_p H(\cdot, Du_\nu) \in L^1.$$

(ii) *The following (in)equalities hold in the sense of distribution:*

$$-\partial_t u_\nu - \nu \Delta u_\nu + H(x, Du_\nu) \leq f(x, m_\nu) \quad \text{in } \Omega,$$

with $u_\nu(T, \cdot) \leq \bar{u}$, and

$$\partial_t m_\nu - \nu \Delta m_\nu - \nabla \cdot (m_\nu D_p(x, Du_\nu)) = 0 \quad \text{in } \Omega \text{ with } m_\nu(0) = \bar{m}.$$

(iii) We have

$$\iint_{\Omega} m_{\nu} f(x, m_{\nu}) + H^*(x, D_p H(x, Du_{\nu})) dx dt + \int_{\mathbb{T}^d} m_{\nu}(T, x) \bar{u}(x) - \bar{m}(x) u_{\nu}(0, x) dx = 0.$$

Here the last term is well-defined due to Lemma 5.1 [15].

Theorem 3.2. *Assume (H1)–(H3) and let $\nu \geq 0$. The problem (2.6) has at least one solution in $\mathcal{K}_{2,\nu}$. Let $(m_{\nu}, w_{\nu}) \in \mathcal{K}_{1,\nu}$ be the minimizer of (2.4) and $(u_{\nu}, \alpha_{\nu}) \in \mathcal{K}_{2,\nu}$ be a minimizer of (2.6). Then:*

- (1) (u_{ν}, m_{ν}) is a weak solution to (1.1), and $w_{\nu} = -m_{\nu} D_p H(\cdot, Du_{\nu})$ and $\alpha_{\nu} = f(\cdot, m_{\nu})$ almost everywhere.
- (2) Conversely, any weak solution (u_{ν}, m_{ν}) of (1.1) is such that the pair $(m_{\nu}, -m_{\nu} D_p H(\cdot, Du_{\nu}))$ is the minimizer of (2.4) and $(u_{\nu}, f(\cdot, m_{\nu}))$ is a minimizer of (2.6).
- (3) The solution is unique in the following sense: if (u'_{ν}, m'_{ν}) is another weak solution to (1.1), then $m_{\nu} = m'_{\nu}$ a.e. and $u_{\nu} = u'_{\nu}$ in $\{m_{\nu} > 0\}$.

The theorem is proved in [15] with an extra assumption that \bar{m} is strictly positive. The assumption is only used in their Proposition 5.4 to show the existence of a solution to the optimization problem (2.6). We use a uniform lower bound of u_{ν} and a slightly different test function to remove the constraint. We state the result below.

Proposition 3.3. *Under the assumptions (H1)–(H3), for each $\nu \geq 0$, the problem (2.6) has at least one solution $(u_{\nu}, \alpha_{\nu}) \in \mathcal{K}_{2,\nu}$ which is uniformly bounded from below by a constant depending only on the assumptions for all $\nu \in [0, 1]$.*

Moreover, there exists $C > 0$ depending only on the assumptions such that

$$\sup_{\nu \in [0,1]} \|u_{\nu}(0, \cdot)\|_{L^1(\mathbb{T}^d)} + \|Du_{\nu}\|_{L^r(\Omega)} + \|\alpha_{\nu}\|_{L^{q'}(\Omega)} \leq C. \quad (3.1)$$

Remark 3.1. *Let us comment that here $\|u_{\nu}(0, \cdot)\|_{L^1(\mathbb{T}^d)}$ is well-defined due to Lemma 5.1 [15] (u_{ν} has a “trace” in a weak sense).*

Proof. The proof follows largely the one of Proposition 5.4 [15].

Step 1. From the first paragraph of their argument, there exists a minimizing sequence $(u_{\nu}^n, \alpha_{\nu}^n) \in \mathcal{K}_{2,\nu}$ for problem (2.6) such that $u_{\nu}^n, \alpha_{\nu}^n$ are continuous, $\alpha_{\nu}^n \geq 0$, and u_{ν}^n is a viscosity solution to

$$-\partial_t u_{\nu}^n - \nu \Delta u_{\nu}^n + H(x, Du_{\nu}^n) = \alpha_{\nu}^n, \quad u_{\nu}^n(T, \cdot) = \bar{u}. \quad (3.2)$$

Since H is convex, this equality also holds in the sense of distribution [37]. Moreover, since $\alpha_{\nu}^n \geq 0$, comparing the solution with $-\|\bar{u}\|_{\infty} - \|H(\cdot, 0)\|_{\infty}(T - t)$ yields

$$u_{\nu}^n(t, x) \geq -\|\bar{u}\|_{\infty} - \|H(\cdot, 0)\|_{\infty}(T - t) \quad \text{in } \Omega. \quad (3.3)$$

It is clear that the right-hand side is independent of ν .

Step 2. Next, we show some bounds for α_{ν}^n and u_{ν}^n that are uniform in n and ν . We integrate (3.2) against $\bar{m} + 1$ on Ω (instead of \bar{m} that was used in [15]) to get

$$\begin{aligned} \int_{\mathbb{T}^d} u_{\nu}^n(0, x) (\bar{m}(x) + 1) dx + \iint_{\Omega} \nu D\bar{m} Du_{\nu}^n + (\bar{m} + 1) H(x, Du_{\nu}^n) dx dt \\ = \iint_{\Omega} (\bar{m}(x) + 1) \alpha_{\nu}^n(t, x) dx dt + \int_{\mathbb{T}^d} \bar{u} (\bar{m} + 1) dx. \end{aligned}$$

By (H2) and (H3) ($\bar{m}, \bar{u} \in \mathcal{C}^1$), there exists $C \geq 1$ depending only on (H3) and C_0 such that

$$\begin{aligned} \int_{\mathbb{T}^d} u_\nu^n(0, x) dx - C\nu \iint_{\Omega} |Du_\nu^n| dx dt + C^{-1} \iint_{\Omega} |Du_\nu^n|^r dx dt \\ \leq - \int_{\mathbb{T}^d} u_\nu^n(0, x) \bar{m}(x) dx + C \iint_{\Omega} |\alpha_\nu^n| dx dt + C. \end{aligned} \quad (3.4)$$

Then, we use (3.3) to get

$$\int_{\mathbb{T}^d} u_\nu^n(0, x) dx \geq \int_{\mathbb{T}^d} |u_\nu^n(0, x)| dx - C$$

for some C depending only on $\|\bar{u}\|_\infty, \|H(\cdot, 0)\|_\infty$ and T . Next, using that $r > 1$ and $q' > 1$, we get for any $\varepsilon > 0$ there is C_ε satisfying

$$\begin{aligned} \iint_{\Omega} |Du_\nu^n| dx dt &\leq \varepsilon \iint_{\Omega} |Du_\nu^n|^r dx dt + C_\varepsilon, \\ \iint_{\Omega} |\alpha_\nu^n| dx dt &\leq \varepsilon \iint_{\Omega} |\alpha_\nu^n|^{q'} dx dt + C_\varepsilon. \end{aligned}$$

Applying these in (3.4) and taking ε to be sufficiently small, we obtain for some C independent of n and $\nu \in [0, 1]$,

$$\begin{aligned} \int_{\mathbb{T}^d} |u_\nu^n(0, x)| dx + (2C)^{-1} \iint_{\Omega} |Du_\nu^n|^r dx dt + (2C_0)^{-1} \iint_{\Omega} |\alpha_\nu^n|^{q'} dx dt \\ \leq - \int_{\mathbb{T}^d} u_\nu^n(0, x) \bar{m}(x) dx + C_0^{-1} \iint_{\Omega} |\alpha_\nu^n|^{q'} dx dt + C \end{aligned}$$

where $C_0 \geq 1$ is from (2.3). It follows from (2.3) and the definition of \mathcal{A} that

$$\begin{aligned} \int_{\mathbb{T}^d} |u_\nu^n(0, x)| dx + (2C)^{-1} \iint_{\Omega} |Du_\nu^n|^r dx dt + (2C_0)^{-1} \iint_{\Omega} |\alpha_\nu^n|^{q'} dx dt \\ \leq - \int_{\mathbb{T}^d} u_\nu^n(0, x) \bar{m}(x) dx + C_0^{-1} \iint_{\Omega} F^*(x, \alpha_\nu^n) dx dt + C_0 + C \leq \mathcal{A}(u_\nu^n, \alpha_\nu^n) + C'. \end{aligned}$$

Step 3. Since (u_ν^n, α_ν^n) is a minimizing sequence, for all n sufficiently large and $\bar{\alpha}_\nu := -\partial_t \bar{u} - \nu \Delta \bar{u} + H(x, D\bar{u})$ we obtain

$$\mathcal{A}(u_\nu^n, \alpha_\nu^n) \leq \sup_{\nu \in [0, 1]} \inf_{(u, \alpha) \in \mathcal{K}_{2, \nu}} \mathcal{A}(u, \alpha) + 1 \leq \sup_{\nu \in [0, 1]} \mathcal{A}(\bar{u}, \bar{\alpha}_\nu) + 1,$$

which is finite. So there exists $C > 0$ independent of n and $\nu \in [0, 1]$ such that for all large n ,

$$\int_{\mathbb{T}^d} |u_\nu^n(0, x)| dx + \iint_{\Omega} |Du_\nu^n|^r dx dt + \iint_{\Omega} |\alpha_\nu^n|^{q'} dx dt \leq C. \quad (3.5)$$

With the uniform bound, one can take weak limit of (u_ν^n, α_ν^n) along a subsequence of $n \rightarrow \infty$. Following Step 2 and Step 3 in Proposition 5.4 [15], we obtain that the weak limit (u_ν, α_ν) is a minimizer of the optimization problem (2.6). Finally, (3.1) follows from (3.5). \square

The qualitative vanishing viscosity limit is studied in [15, Theorem 6.5]. We state it below.

Theorem 3.4 ([15]). *Let (u_ν, m_ν) be a weak solution to (1.1) with $\nu > 0$. Then m_ν converges strongly to m in $L^q(\Omega)$ as $\nu \rightarrow 0$, and u_ν converges weakly up to a subsequence to u in $L^\gamma(\Omega)$, where the pair (u, m) is a weak solution to (1.1) with $\nu = 0$.*

4. VANISHING VISCOSITY LIMIT – LOCAL COUPLING

To study the convergence rate of vanishing viscosity limit of mean field games, we further need the following regularity and coercivity assumptions.

(H4) (Regularity condition) There exist $C_0, C_1, C_2 \geq 0$ such that for all $x, y \in \mathbb{T}^d$ and $m \geq 0$,

$$|F(x + y, m) - F(x, m)| \leq C_0(1 + m^q)|y|,$$

and for any $p \in \mathbb{R}^d$ and $\sigma \in (0, 1]$,

$$\sup_{z \in B_\sigma} H\left(x + z, \frac{p}{1 + C_1\sigma}\right) \leq \frac{H(x, p)}{1 + C_1\sigma} + C_2\sigma. \quad (4.1)$$

(H5-1) (Coercivity condition on H) There are $J_1, J_1^* : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $c_0 > 0$ such that for all $p, \xi \in \mathbb{R}^d$ we have

$$H(x, p) + H^*(x, \xi) - p \cdot \xi \geq c_0 |J_1(x, p) - J_1^*(x, \xi)|^2. \quad (4.2)$$

(H5-2) (Coercivity condition on F) There are $J_2, J_2^* : \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R}$ and $c_0 > 0$ such that for all $m, \alpha \in \mathbb{R}$ we have

$$F(x, m) + F^*(x, \alpha) - m\alpha \geq c_0 |J_2(x, m) - J_2^*(x, \alpha)|^2. \quad (4.3)$$

Remark 4.1. 1. If $H(x, p)$ is only a function of p , then we can take $C_1 = C_2 = 0$ in (H4) and (4.1) is trivial.

The condition (4.1) on H is stronger than the one on F , but it is still satisfied by a large class of Hamiltonian. For example, if $H(x, p) = h_1(x)|p|^r + h_2(x)$ with two Lipschitz continuous functions $h_1, h_2 : \mathbb{T}^d \rightarrow \mathbb{R}$ and $h_1 \geq 0$, then (4.1) holds with $C_1 := \|Dh_1\|_\infty / (r - 1)$ and $C_2 := \|Dh_2\|_\infty$.

2. Note that $F^*(x, \alpha) = 0$ for $\alpha \leq 0$. For $\alpha \geq 0$, there exists a unique $m_\alpha \geq 0$ such that

$$F^*(x, \alpha) = \sup_{m \geq 0} \{\alpha m - F(x, m)\} = \alpha m_\alpha - F(x, m_\alpha).$$

By (2.2), $m_\alpha \geq 0$ satisfies

$$m_\alpha^q \leq C\alpha^{q'} + C \quad \text{for some } C > 0. \quad (4.4)$$

So it follows from (H4) that for all $x, y \in \mathbb{T}^d$ and $\alpha \geq 0$,

$$|F^*(x + y, \alpha) - F^*(x, \alpha)| \leq C(1 + \alpha^{q'})|y| \quad \text{for some } C > 0. \quad (4.5)$$

It is not hard to see that F^* is increasing in α . Indeed for any $\sigma \geq 0$,

$$F^*(x, \alpha + \sigma) \geq (\alpha + \sigma)m_\alpha - F(x, m_\alpha) \geq F^*(x, \alpha). \quad (4.6)$$

Moreover, by (4.4), for any $\sigma \geq 0$,

$$F^*(x, \alpha) \geq F^*(x, \alpha + \sigma) - \sigma m_{\alpha + \sigma} \geq F^*(x, \alpha + \sigma) - C\sigma(1 + (\alpha + \sigma)^{q'/q}), \quad (4.7)$$

where $m_{\alpha + \sigma}$ is such that $F^*(x, \alpha + \sigma) = \alpha m_{\alpha + \sigma} - F(x, m_{\alpha + \sigma})$

3. Let $\tau : \mathbb{T}^d \rightarrow (0, \infty)$ and $h : \mathbb{T}^d \rightarrow \mathbb{R}$ be continuous. If $H(x, p) = \frac{1}{r}|\tau(x)p|^r + h(x)$ for some $r > 1$, then $H^*(x, \xi) = \frac{1}{r'}|\xi/\tau(x)|^{r'} - h(x)$. In this case, (4.2) holds with

$$J_1(x, p) = \tau(x)^{r/2}|p|^{r/2-1}p, \quad J_1^*(x, \xi) = \tau(x)^{-r'/2}|\xi|^{r'/2-1}\xi \quad \text{and} \quad c_0 = (\max\{r, r'\})^{-1}.$$

If $F(x, m) = \frac{1}{q}(\tau(x)m)^q$ for some $q > 1$, then $F^*(x, \alpha) = \frac{1}{q'}(\alpha/\tau(x))^{q'}$, and (4.3) holds with

$$J_2(x, m) = \tau(x)^{q/2}m^{q/2}, \quad J_2^*(x, \alpha) = \tau(x)^{-q'/2}\alpha^{q'/2} \quad \text{and} \quad c_0 = (\max\{q, q'\})^{-1}.$$

We only prove (4.3) with this selection of functions in the appendix. The proof for (4.2) is almost the same.

Finally, let us comment that in the above examples the value of c_0 is optimal. Indeed, if $\tau \equiv 1, q = q' = 2$, it is easy to see that $c_0 \leq \frac{1}{2}$. The optimality can also be seen in the proof.

The following lemma is a consequence of the coercivity condition. The proof is identical to the one of Lemma 3.2 [54], and we skip it.

Lemma 4.1. *Assume (H1)–(H4), and for $\nu \geq 0$, let $(u, \alpha) \in \mathcal{K}_{2,\nu}$ and $(m, w) \in \mathcal{K}_{1,\nu}$. If (H5-1) holds, we have*

$$\mathcal{A}(u, \alpha) + \mathcal{B}(m, w) \geq c_0 \|m^{1/2}(J_1(\cdot, Du) - J_1^*(\cdot, -w/m))\|_{L^2(\Omega)}^2. \quad (4.8)$$

And if (H5-2) holds, we have

$$\mathcal{A}(u, \alpha) + \mathcal{B}(m, w) \geq c_0 \|J_2(\cdot, m) - J_2^*(\cdot, \alpha)\|_{L^2(\Omega)}^2. \quad (4.9)$$

For $\nu \geq 0$, let $(m_\nu, w_\nu) \in \mathcal{K}_{1,\nu}$ be the minimizer of (2.4) and $(u_\nu, \alpha_\nu) \in \mathcal{K}_{2,\nu}$ be a minimizer of (2.6). Theorem 3.2 and Theorem 2.1 yield that (u_ν, m_ν) is a weak solution to (1.1), and

$$\mathcal{A}(u_\nu, \alpha_\nu) + \mathcal{B}(m_\nu, w_\nu) = 0. \quad (4.10)$$

The following result is a direct consequence of Lemma 4.1 and (4.10).

Corollary 4.2. *We have for any $\nu \geq 0$ and for a.e. $(t, x) \in \Omega$ such that*

$$J_1^*(x, -w_\nu/m_\nu) = J_1(x, Du_\nu) \quad \text{and} \quad J_2^*(x, \alpha_\nu) = J_2(x, m_\nu). \quad (4.11)$$

Now we prove the main theorem of the section.

Theorem 4.3. *Assume (H1)–(H4), and let (m_ν, α_ν) be a weak solution to (1.1) with $\nu \geq 0$. Write $(u, \alpha, m, w) = (u_0, \alpha_0, m_0, w_0)$, and*

$$\beta := \max \left\{ \frac{q'r'}{q' + r'}, 1 \right\} (d+1) - d \geq 1. \quad (4.12)$$

Then if (H5-1) holds, there exists $C > 0$ such that for all $\nu \in [0, 1]$ we have

$$\iint_{\Omega} |J_1(\cdot, Du_\nu) - J_1(\cdot, Du)|^2 m \, dxdt \leq C\nu^{\frac{1}{1+\beta}},$$

and if (H5-2) holds, for all $\nu \in [0, 1]$ we have

$$\iint_{\Omega} |J_2(\cdot, m_\nu) - J_2(\cdot, m)|^2 \, dxdt \leq C\nu^{\frac{1}{1+\beta}}.$$

Proof. We use the optimization structure of (1.1) described in Theorem 3.2.

Step 1. By Theorem 2.1 and the proof of Proposition 3.3, we can take a minimizing sequence $(u^n, \alpha^n) \in \mathcal{K}_{2,0}$ such that $\alpha^n \geq 0$, and

$$-\partial_t u^n + H(x, Du^n) = \alpha^n, \quad u^n(T, \cdot) = \bar{u} \quad (4.13)$$

in the sense of distribution, and

$$\mathcal{A}(u^n, \alpha^n) \leq \mathcal{A}(u, \alpha) + 1/n.$$

Due to (3.1), there exists $C > 0$ such that for all n sufficiently large we have

$$\|Du^n\|_{L^r(\Omega)} + \|\alpha^n\|_{L^{q'}(\Omega)} \leq C. \quad (4.14)$$

Step 2. The goal is to show that

$$\mathcal{A}(u_\nu, \alpha_\nu) \leq \mathcal{A}(u, \alpha) + C\nu^{\frac{1}{1+\beta}}. \quad (4.15)$$

Below we construct $(u_\varepsilon^n, \alpha_\varepsilon^n)$ from (u^n, α^n) that can be used as a candidate for the optimization problem (2.6).

Let $\varphi : \mathbb{R}^d \rightarrow [0, \infty)$ be a mollifier i.e., φ is smooth, compactly supported in B_1 , and $\int_{\mathbb{R}^d} \varphi(x) dx = 1$. Take $\varepsilon := \nu^{1/(1+\beta)}$ and set $\varphi_\varepsilon := \varepsilon^{-d} \varphi(\cdot/\varepsilon)$. For C_1, C_2 from (H4), we define

$$u_\varepsilon^n := \frac{\varphi_\varepsilon * u^n}{1 + C_1 \varepsilon} - (1 + C_1) \varepsilon \|\bar{u}\|_{C^1} + C_2 \varepsilon (t - T). \quad (4.16)$$

It is direct to see from the definition that $u_\varepsilon^n(T, \cdot) \leq \bar{u}$ as φ_ε is supported in B_ε .

Step 3. Because $H(x, \cdot)$ is convex, Jensen's inequality yields for each fixed $x_0 \in \mathbb{T}^d$,

$$H\left(x_0, D\left(\frac{\varphi_\varepsilon * u^n}{1 + C_1 \varepsilon}\right)(t, x)\right) \leq \varphi_\varepsilon * H\left(x_0, D\left(\frac{u^n}{1 + C_1 \varepsilon}\right)\right)(t, x).$$

By (4.1) with ε in place of σ , we obtain

$$H(x, Du_\varepsilon^n(t, x)) \leq \frac{\varphi_\varepsilon * H(x, Du^n)(t, x)}{1 + C_1 \varepsilon} + C_2 \varepsilon.$$

We note that if H is independent of x , then this holds with $C_1 = C_2 = 0$. We get from (4.13) that

$$-\partial_t u_\varepsilon^n - \nu \Delta u_\varepsilon^n + H(x, Du_\varepsilon^n) \leq \frac{\varphi_\varepsilon * \alpha^n}{1 + C_1 \varepsilon} - \frac{\nu}{1 + C_1 \varepsilon} \Delta \varphi_\varepsilon * u^n =: \alpha_\varepsilon^n. \quad (4.17)$$

In particular, we have $(u_\varepsilon^n, \alpha_\varepsilon^n) \in \mathcal{A}_{2,\nu}$. Consequently, as (u_ν, α_ν) is a minimizer of (2.6) over $\mathcal{K}_{2,\nu}$, we obtain

$$\mathcal{A}(u_\nu, \alpha_\nu) \leq \mathcal{A}(u_\varepsilon^n, \alpha_\varepsilon^n).$$

It follows from the definition of ε that $\nu \varepsilon^{-\beta} \leq 1$. We claim that

$$\mathcal{A}(u_\nu, \alpha_\nu) \leq \mathcal{A}(u_\varepsilon^n, \alpha_\varepsilon^n) \leq \mathcal{A}(u^n, \alpha^n) + C\nu \varepsilon^{-\beta} + C\varepsilon.$$

We postpone the proof to Lemma 4.4 below. As a consequence of the claim, passing $n \rightarrow \infty$ yields (4.15).

Step 4. Now, we combine (4.15) with (4.10) to get

$$\mathcal{A}(u_\nu, \alpha_\nu) + \mathcal{B}(m, w) \leq \mathcal{A}(u, \alpha) + \mathcal{B}(m, w) + C\nu^{\frac{1}{1+\beta}} = C\nu^{\frac{1}{1+\beta}}.$$

Then using (4.8) and (H5-1) yields

$$c_0 \|m^{1/2}(J_1(\cdot, Du_\nu) - J_1^*(\cdot, -w/m))\|_{L^2(\Omega)}^2 \leq \mathcal{A}(u_\nu, \alpha_\nu) + \mathcal{B}(m, w) \leq C\nu^{\frac{1}{1+\beta}}.$$

Using instead (4.9) and (H5-2) yields

$$c_0 \|J_2(\cdot, m) - J_2^*(\cdot, \alpha_\nu)\|_{L^2(\Omega)}^2 \leq \mathcal{A}(u_\nu, \alpha_\nu) + \mathcal{B}(m, w) \leq C\nu^{\frac{1}{1+\beta}}.$$

Finally (4.11) yields the conclusion. \square

Remark 4.2. Suppose that $J_2(\cdot, m) \equiv m^{q/2}$. We obtain from the second conclusion of theorem that when $q \geq 2$,

$$\begin{aligned} \iint_{\Omega} |m_\nu - m|^2 dxdt &\leq C \left(\iint_{\Omega} |m_\nu - m|^q dxdt \right)^{2/q} \\ &\leq C \left(\iint_{\Omega} |m_\nu^{q/2} - m^{q/2}|^2 dxdt \right)^{2/q} \leq C\nu^{\frac{2}{q(1+\beta)}}. \end{aligned}$$

Lemma 4.4. Suppose $\nu\varepsilon^{-\beta} \leq 1$. Then

$$\mathcal{A}(u_\varepsilon^n, \alpha_\varepsilon^n) \leq \mathcal{A}(u^n, \alpha^n) + C\nu\varepsilon^{-\beta} + C\varepsilon.$$

Proof. In view of (4.14), Young's convolution inequality yields if $\frac{q'r'}{q'+r'} \geq 1$ (then $r \leq q'$),

$$\|\Delta\varphi_\varepsilon * u^n\|_{L^{q'}(\Omega)} \leq \|D\varphi_\varepsilon\|_{L^{\beta_1}(\Omega)} \|Du^n\|_{L^r(\Omega)} \leq C\varepsilon^{-\beta_1(d+1)+d} \quad \text{with } \beta_1 := \frac{q'r'}{q'+r'} \geq 1,$$

and if $\frac{q'r'}{q'+r'} < 1$ (then $r > q'$),

$$\|\Delta\varphi_\varepsilon * u^n\|_{L^{q'}(\Omega)} \leq C\|\Delta\varphi_\varepsilon * u^n\|_{L^r(\Omega)} \leq C\|D\varphi_\varepsilon\|_{L^1(\Omega)} \|Du^n\|_{L^r(\Omega)} \leq C\varepsilon^{-1}.$$

Thus, overall, we get for $X_\varepsilon := -\nu\Delta\varphi_\varepsilon * u^n$,

$$\|X_\varepsilon\|_{L^{q'}(\Omega)} \leq C\nu\varepsilon^{-\beta} \quad \text{with } \beta = \max\{\beta_1, 1\}(d+1) - d. \quad (4.18)$$

In view of (4.17),

$$\alpha_\varepsilon^n - \varphi_\varepsilon * \alpha^n / (1 + C_1\varepsilon) \leq (X_\varepsilon)_+.$$

Using (4.6) and (4.7) yields

$$\begin{aligned} \iint_{\Omega} F^*(x, \alpha_\varepsilon^n) dxdt &\leq \iint_{\Omega} F^*(x, \frac{\varphi_\varepsilon * \alpha^n}{1 + C_1\varepsilon}) + C(X_\varepsilon)_+(1 + (\alpha_\varepsilon^n)^{q'/q}) dxdt \\ &\leq \iint_{\Omega} F^*(x, \varphi_\varepsilon * \alpha^n) + C(X_\varepsilon)_+(1 + (\varphi_\varepsilon * \alpha^n + (X_\varepsilon)_+)^{\frac{q'}{q}}) dxdt \quad (4.19) \\ &\leq \iint_{\Omega} F^*(x, \varphi_\varepsilon * \alpha^n) + C(X_\varepsilon)_+^{q'} + C(X_\varepsilon)_+(1 + (\varphi_\varepsilon * \alpha^n)^{\frac{q'}{q}}) dxdt. \end{aligned}$$

It follows from (4.18), Hölder's and Young's convolution inequality that

$$\begin{aligned} \iint_{\Omega} (X_\varepsilon)_+^{q'} + (X_\varepsilon)_+(1 + (\varphi_\varepsilon * \alpha^n)^{\frac{q'}{q}}) dxdt \\ \leq \|X_\varepsilon\|_{L^{q'}(\Omega)}^{q'} + \|X_\varepsilon\|_{L^{q'}(\Omega)} \left(\iint_{\Omega} 1 + (\varphi_\varepsilon * \alpha^n)^{q'} dxdt \right)^{1/q} \\ \leq C\nu^{q'}\varepsilon^{-q'\beta} + C\nu\varepsilon^{-\beta} \left(1 + \|\alpha^n\|_{L^{q'}(\Omega)}^{q'/q} \right) \leq C\nu\varepsilon^{-\beta} \end{aligned}$$

where the last inequality is due to (4.14) and $\nu\varepsilon^{-\beta} \leq 1$. Since $F^*(x, \cdot)$ is convex, Jensen's inequality and (4.5) show

$$\begin{aligned} \iint_{\Omega} F^*(x, \varphi_\varepsilon * \alpha^n) dxdt &\leq \iint_{\Omega} \varphi_\varepsilon * F^*(x, \alpha^n) dxdt + C\varepsilon(1 + (\alpha^n)^{q'}) dxdt \\ &\leq \iint_{\Omega} F^*(x, \alpha^n) dxdt + C\varepsilon, \end{aligned}$$

where we applied (4.14) again. Using these in (4.19) yields

$$\iint_{\Omega} F^*(x, \alpha_\varepsilon^n) dx dt \leq \iint_{\Omega} F^*(x, \alpha^n) dx dt + C\nu\varepsilon^{-\beta} + C\varepsilon. \quad (4.20)$$

Note that, since $\bar{m} \in \mathcal{C}^1$, from (4.16) and (3.5) we get

$$\begin{aligned} \int_{\mathbb{T}^d} u^n(0, x)\bar{m}(x) - u_\varepsilon^n(0, x)\bar{m}(x) dx &\leq C\varepsilon + \int_{\mathbb{T}^d} u^n(0, x)\bar{m}(x) - u^n(0, x)\frac{\varphi_\varepsilon * \bar{m}(x)}{1 + C_1\varepsilon} dx \\ &\leq C\varepsilon \left(1 + \int_{\mathbb{T}^d} |u^n(0, x)| dx\right) \leq C\varepsilon. \end{aligned} \quad (4.21)$$

Finally, we recall the definition of \mathcal{A} and apply (4.20) and (4.21) to get

$$\begin{aligned} \mathcal{A}(u_\varepsilon^n, \alpha_\varepsilon^n) &= \iint_{\Omega} F^*(x, \alpha_\varepsilon^n) dx dt - \int_{\mathbb{T}^d} u_\varepsilon^n(0, \cdot)\bar{m} dx \\ &\leq \iint_{\Omega} F^*(x, \alpha^n) dx dt - \int_{\mathbb{T}^d} u^n(0, \cdot)\bar{m} dx + C\nu\varepsilon^{-\beta} + C\varepsilon \\ &= \mathcal{A}(u^n, \alpha^n) + C\nu\varepsilon^{-\beta} + C\varepsilon. \end{aligned}$$

□

5. CONVERGENCE OF CLASSICAL SOLUTIONS – LOCAL COUPLING

In this section, we prove the convergence of u_ν under some extra conditions (see (H6)–(H8) below). We will assume that (u_ν, m_ν) with $\nu > 0$ are classical solutions. Indeed, [16] showed that the solutions are smooth when Hamiltonians are purely quadratic. We also refer readers to [29, 28] for results about classical solutions.

We will present two results for quadratic Hamiltonians (so $q = 2$). The first one is for $r > 2$ and we use the conditions (H6)(H7) below (by (H6) we mean (H6-1)–(H6-3)), while the second one is for $r = 2$ and $d \leq 3$, and we further need (H8).

(H6-1) $\bar{m} \in W^{2,\infty}(\mathbb{T}^d)$.

(H6-2) $H(x, p)$ is twice continuously differentiable in both the variables and for some $C > 0$,

$$|D_x^2 H(x, p)| \leq C|p|^r + C \quad \text{for all } (x, p) \in \mathbb{T}^d \times \mathbb{R}^d.$$

(H6-3) There exist $C \geq 1$ such that for all $x, y \in \mathbb{T}^d$ and $m, m' \geq 0$,

$$|f(x, m) - f(y, m)| \leq Cm|x - y|,$$

and

$$(f(x, m) - f(x, m'))(m - m') \geq |m - m'|^2/C. \quad (5.1)$$

(H7) (Conditions on the coupling) (H1) holds with $q = 2$, and for some $C > 0$,

$$f_m(x, m) \leq C \quad \text{for } (x, m) \in \mathbb{T}^d \times [0, \infty).$$

(H8) (Strict convex Hamiltonian) For some $C \geq 1$.

$$C^{-1}I_d \leq H_{pp}(x, p) \leq CI_d \quad \text{for all } (x, p) \in \mathbb{T}^d \times \mathbb{R}^d,$$

where I_d denotes the identity matrix.

Remark 5.1. Condition (H6) is assumed in [31]. We are going to apply their regularity results. (H8) implies $r = 2$ in (H2).

We start with the following lemma which says that (5.1) implies (H5-2) with $J_2(x, m) = m$, and (H8) implies (H5-1) with $J_1(x, p) = p$.

Lemma 5.1. *Assume (H1)(H6). Then (H5-2) holds with $J_2(x, m) = m$ and $J_2^*(x, \alpha)$ defined such that $f(x, J_2^*(x, \alpha)) = \alpha$.*

If further assuming $H_{pp}(x, p) \geq cI_d$ for some $c > 0$ uniformly in $\mathbb{T}^d \times \mathbb{R}^d$, then (H5-1) holds with $J_1(x, p) = p$ and $J_1^(x, \xi)$ defined such that $H_p(x, J_1^*(x, \xi)) = \xi$.*

Here J_2^ is well-defined as $f(x, \alpha)$ is strictly increasing in α . Similarly, J_1^* is well-defined as $H(x, \cdot)$ is C^2 and is uniformly strictly convex in p , and the map $H_p(x, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is one-to-one.*

Proof. Let us fix $x \in \mathbb{T}^d$. In view of Remark 4.1.2, it suffices to consider $m, \alpha > 0$. Take m_α such that $f(x, m_\alpha) = \alpha$. The definition of F^* yields

$$F(x, m) + F^*(x, \alpha) - m\alpha \geq F(x, m) - F(x, m_\alpha) + (m_\alpha - m)\alpha. \quad (5.2)$$

Since F is strictly convex by (5.1), and $F_m(x, m) = f(x, m)$, we have

$$F(x, m) - F(x, m_\alpha) \geq f(x, m_\alpha)(m - m_\alpha) + \frac{c}{2}|m - m_\alpha|^2.$$

Then, using $f(x, m_\alpha) = \alpha$, we get from (5.2) that

$$F(x, m) + F^*(x, \alpha) - m\alpha \geq c(m - J_2^*(x, \alpha))^2/2$$

where $J_2^*(x, \alpha) := m_\alpha$. We proved the first claim.

The proof for the second claim is identical. \square

In the next lemma, we collect and prove some ν -uniform estimates.

Lemma 5.2. *Assume (H2)–(H4)(H6)(H7), and suppose that (m_ν, α_ν) with $\nu \in [0, 1]$ are weak solutions to (1.1). Then there exists $C > 0$ such that for all $\nu \in [0, 1]$,*

$$\|Dm_\nu\|_{L^2(\Omega)} \leq C \quad \text{and} \quad (5.3)$$

$$\|m_\nu\|_{L^\eta(\Omega)} + \|u_\nu\|_{L^\delta(\Omega)} \leq C,$$

where $\eta := \frac{2(d+1)}{d}$ and

$$\delta := \frac{r\eta(d+1)}{d - r(\eta - 1)} \quad \text{if } \eta < 1 + \frac{d}{r} \quad \text{and} \quad \delta := \infty \quad \text{if } \eta > 1 + \frac{d}{r}. \quad (5.4)$$

When $\eta = 1 + \frac{d}{r}$, $\delta > 0$ can be an arbitrarily large constant, and in this case, the constant C depends also on the choice of δ .

If further assuming $\inf_{\mathbb{T}^d \times \mathbb{R}^d} H_{pp}(x, p) \geq cI_d$ for some $c > 0$, then we have

$$\|m_\nu^{1/2} D^2 u_\nu\|_{L^2(\Omega)} \leq C \quad \text{uniformly in } \nu \in [0, 1] \text{ for some } C > 0. \quad (5.5)$$

Proof. By Lemma 5.1, $J_1(x, \cdot)$ is an identity map. Then (5.3) follows from [31, Proposition 4.3] and (H6)(H7). In [31], the authors only considered the case of $\nu = 0$ and assumed (H5-1) with J_1, J_1^* independent of x . However, this assumption is not needed to obtain only (5.3). Also it is not hard to see that their argument generalizes to all $\nu \in [0, 1]$ and the constant is independent of ν .

Note that $m_\nu \geq 0$ and $\int_{\mathbb{T}^d} m_\nu(t, \cdot) = 1$. Thus it follows from (5.3) and the Sobolev type embedding theorem ([21, Proposition 3.1]) that for some dimensional constant $C > 0$,

$$\iint_{\Omega} m_\nu^\eta(t, x) dx dt \leq C \left(\iint_{\Omega} |Dm_\nu(t, x)|^2 dx dt \right) \left(\operatorname{ess\,sup}_{t \in (0, T)} \int_{\mathbb{T}^d} m_\nu(t, x) dx \right)^{2/d},$$

which is uniformly finite in ν by (5.3).

Next, due to (H1) and $q = 2$, $f(x, m_\nu) \in L^\eta(\Omega)$ by (5.3). Moreover, Proposition 3.3 yields that u_ν is bounded from below. So we can apply Theorem 3.3 in [15] to get for some C independent of $\nu \in [0, 1]$,

$$\|u_\nu\|_{L^\delta(\Omega)} \leq C \quad \text{with } \delta \text{ given in (5.4).}$$

It is not hard to see that when $\eta = 1 + \frac{d}{r}$, $\delta > 0$ can be arbitrary and the constant C depends also on δ .

The last claim follows from [31] and the second claim of Lemma 5.1. \square

Now we are ready to prove our main theorem of the section.

Theorem 5.3. *Assume (H2)–(H4)(H6)(H7) and*

$$r \geq \max \left\{ 2 + \frac{d}{d+1}, \frac{d^2}{2d+3} \right\}. \quad (5.6)$$

Suppose that (u_ν, m_ν) with $\nu \in (0, 1]$ are classical solutions to (1.1), and (u, m) is a weak solution to (1.1) with $\nu = 0$. Then there exists $C > 0$ such that for all $\nu \in (0, 1]$,

$$\iint_{\Omega} |u_\nu - u|^2 m dx dt \leq C \nu^{\frac{1}{2(1+\beta)}},$$

where β is given in (4.12).

Proof. The proof is split into four steps.

Step 1. For any $\nu' \in (0, \nu)$, define

$$U(t, x) := u_\nu(t, x) - u_{\nu'}(t, x).$$

Later we will pass $\nu' \rightarrow 0$. From the first equation in (1.1), it follows that that

$$-\partial_t U - \nu \Delta u_\nu + \nu' \Delta u_{\nu'} + H(x, Du_\nu) - H(x, Du_{\nu'}) = f(x, m_\nu) - f(x, m_{\nu'}). \quad (5.7)$$

Since (u_ν, m_ν) and $(u_{\nu'}, m_{\nu'})$ are classical solutions, we can multiply (5.7) by $-2Um_{\nu'}$, and use U^2 as the test function against the equation of $m_{\nu'}$. Then adding them up and integrating from t to T for some $t \in [0, T)$, we find

$$\begin{aligned} & \int_{\mathbb{T}^d} U^2 m_{\nu'} dx \Big|_t^T - \int_t^T \int_{\mathbb{T}^d} 2(\nu Du_\nu - \nu' Du_{\nu'}) D(Um_{\nu'}) + \nu' Dm_{\nu'} D(U^2) dx dt \\ &= \int_t^T \int_{\mathbb{T}^d} 2(H(x, Du_\nu) - H(x, Du_{\nu'})) U m_{\nu'} - D_p H(x, Du_{\nu'}) D(U^2) m_{\nu'} \\ & \quad - 2(f(x, m_\nu) - f(x, m_{\nu'})) U m_{\nu'} dx dt. \end{aligned} \quad (5.8)$$

Step 2. Since $H(x, p)$ is convex in p ,

$$H(x, Du_\nu) - H(x, Du_{\nu'}) - D_p H(x, Du_{\nu'})(DU) \geq 0.$$

Also using $U(T, \cdot) = 0$, (5.8) yields

$$\begin{aligned} \int_{\mathbb{T}^d} U^2(t, \cdot) m_{\nu'}(t, \cdot) dx &\leq \int_t^T \int_{\mathbb{T}^d} 2(\nu Du_\nu - \nu' Du_{\nu'}) D(Um_{\nu'}) + \nu' Dm_{\nu'} D(U^2) dx dt \\ &\quad + 2 \int_t^T \int_{\mathbb{T}^d} (f(x, m_\nu) - f(x, m_{\nu'})) U m_{\nu'} dx dt. \end{aligned} \quad (5.9)$$

Step 3. Now we estimate each term on the right-hand side of (5.9). Recall η, δ from Lemma 5.2 (with $q' = 2$ since $q = 2$ by (H7)). By Hölder's inequality,

$$\int_t^T \int_{\mathbb{T}^d} (Du_\nu DU) m_{\nu'} dx dt \leq \|Du_\nu\|_{L^r(\Omega)} \|DU\|_{L^r(\Omega)} \|m_{\nu'}\|_{L^{r_1}(\Omega)}$$

where $r_1 := \frac{2}{r-2} \leq \eta$ by (5.6). Thus, Lemma 5.2 and Proposition 3.3 yield

$$\int_t^T \int_{\mathbb{T}^d} (Du_\nu DU) m_{\nu'} dx dt \leq C$$

for some C independent of ν, ν' . Similarly, we have $\int_t^T \int_{\mathbb{T}^d} |Du_{\nu'} DU| m_{\nu'} dx dt \leq C$. Due to Lemma 5.2 and (3.1) again, we get

$$\int_t^T \int_{\mathbb{T}^d} (Du_\nu Dm_{\nu'}) U dx dt \leq \|Du_\nu\|_{L^r(\Omega)} \|Dm_{\nu'}\|_{L^2(\Omega)} \|U\|_{L^{r_2}(\Omega)} \leq C$$

where $r_2 := \frac{2r}{r-2} \leq \delta$ by (5.6). Similarly, $\int_t^T \int_{\mathbb{T}^d} Dm_{\nu'} D(U^2) dx dt$ is uniformly bounded.

As for $\int_t^T \int_{\mathbb{T}^d} (f(x, m_\nu) - f(x, m_{\nu'})) U m_{\nu'} dx dt$, using (H7) yields

$$\begin{aligned} \int_t^T \int_{\mathbb{T}^d} (f(x, m_\nu) - f(x, m_{\nu'})) U m_{\nu'} dx dt &\leq C \iint_{\Omega} |m_\nu - m_{\nu'}| |U| m_{\nu'} dx dt \\ &\leq C \|m_\nu - m_{\nu'}\|_{L^2(\Omega)} \|U\|_{L^\delta(\Omega)} \|m_{\nu'}\|_{L^{r_3}} \end{aligned}$$

where $r_3 := \frac{2\delta}{\delta-2}$. Since $r_3 \leq \eta$ by (5.6), $\|U\|_{L^\delta(\Omega)} \|m_{\nu'}\|_{L^{r_3}} \leq C$ by Lemma 5.2. Thus, Theorem 4.3 and the first part of Lemma 5.1 yield

$$\iint_{\Omega} (f(x, m_\nu) - f(x, m_{\nu'})) U m_{\nu'} dx dt \leq C \nu^{\frac{1}{2(1+\beta)}} \quad \text{with } \beta \text{ from (4.12).}$$

Step 4. Putting these estimates into (5.9) yields

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^d} |u_\nu - u_{\nu'}|^2 m_{\nu'} dx \leq C \nu^{\frac{1}{2(1+\beta)}}.$$

By Theorem 3.4, $m_{\nu'}$ converges strongly to m in $L^2(\Omega)$ as $\nu' \rightarrow 0$. Thus, as $u_\nu - u_{\nu'}$ is uniformly bounded in $L^4(\Omega)$ by Lemma 5.2 and $\delta \geq 4$, we get

$$\liminf_{\nu' \rightarrow 0} \iint_{\Omega} |u_\nu - u_{\nu'}|^2 m_{\nu'} dx dt = \liminf_{\nu' \rightarrow 0} \iint_{\Omega} |u_\nu - u_{\nu'}|^2 m dx dt.$$

Since $u_{\nu'}$ are uniformly bounded in $L^\delta(\Omega)$ and $u_{\nu'}$ converges weakly to u in $L^r(\Omega)$ along a subsequence of $\nu' \rightarrow 0$ by Theorem 3.4, we actually have $u_\nu - u_{\nu'}$ converges weakly to $u_\nu - u$ in $L^\delta(\Omega)$ along a subsequence of $\nu' \rightarrow 0$. Then we show that the functional $v \rightarrow \iint_{\Omega} v^2 m dx dt$

is continuous on $L^\delta(\Omega)$. Let $\eta' = \frac{\eta}{\eta-1}$ be the conjugate of η . Note that for any $v_1, v_2 \in L^\delta(\Omega)$, by Lemma 5.2,

$$\iint_{\Omega} (v_1^2 - v_2^2) m dx dt \leq C \left(\iint_{\Omega} (v_1^2 - v_2^2)_+^{\eta'} dx dt \right)^{1/\eta'} \leq C \left(\iint_{\Omega} (v_1^{2\eta'} - v_2^{2\eta'})_+ dx dt \right)^{1/\eta'}, \quad (5.10)$$

and by (5.3),

$$\delta \geq 2\eta' = 2\eta/(\eta - 1).$$

The right-hand side of (5.10) can be arbitrarily small when v_1 and v_2 are very close in $L^r(\Omega)$. Thus $v \rightarrow \iint_{\Omega} v^2 m dx dt$ is continuous on $L^r(\Omega)$, and it is straightforward that it is also convex. Hence we get

$$\iint_{\Omega} |u_\nu - u|^2 m dx dt \leq \liminf_{\nu' \rightarrow 0} \iint_{\Omega} |u_\nu - u_{\nu'}|^2 m dx dt \leq C\nu^{\frac{1}{2(1+\beta)}},$$

which finishes the proof. \square

Theorem 5.3 fails to cover the KPZ setting (1.2) when $r = 2$ that we are interested in. Below, we prove a convergence result for $d \leq 3$ and $q = r = 2$. The statement is slightly different from the previous theorem that we use weight m_ν instead of m .

Theorem 5.4. *Assume (H2)–(H4)(H6)–(H8), $r = 2$ and $d \leq 3$. Suppose that (u_ν, m_ν) with $\nu \in (0, 1]$ are classical solutions to (1.1), and (u, m) is a weak solution to (1.1) with $\nu = 0$. Then there exists $C > 0$ such that for all $\nu \in (0, 1]$,*

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^d} |u_\nu(t, x) - u(t, x)|^2 m_\nu(t, x) dx \leq C\nu^{\frac{1}{2(1+\beta)}},$$

where β is given in (4.12).

Proof. We break the proof into three steps.

Step 1. Since $d \leq 3$, Lemma 5.2 yields that u_ν is uniformly bounded for all $\nu \in [0, 1]$ in Ω .

For $0 < \nu' < \nu < 1$, let

$$U(t, x) := u_\nu(t, x) - u_{\nu'}(t, x).$$

Then the following holds in the classical sense,

$$-\partial_t U - \nu \Delta u_\nu + \nu' \Delta u_{\nu'} + H(x, Du_\nu) - H(x, Du_{\nu'}) = f(x, m_\nu) - f(x, m_{\nu'}).$$

We multiply the above equality by $-2Um_\nu$, and use U^2 as the test function against the equation of m_ν in (1.1). Adding them up and integrating from t to T for some $t \in [0, T]$ yield

$$\begin{aligned} & \int_{\mathbb{T}^d} U^2 m_\nu dx \Big|_t^T + \iint_{\Omega} 2\nu \Delta u_\nu (Um_\nu) + \nu \nabla m_\nu \nabla (U^2) - 2\nu' \Delta u_{\nu'} (Um_\nu) dx dt \\ &= \iint_{\Omega} 2(H(x, Du_\nu) - H(x, Du_{\nu'})) Um_\nu - D_p H(x, Du_\nu) D(U^2) m_\nu \\ & \quad - 2(f(x, m_\nu) - f(x, m_{\nu'})) Um_\nu dx dt. \end{aligned}$$

Since $U(T, \cdot) \equiv 0$, this simplifies to

$$\begin{aligned} \int_{\mathbb{T}^d} U^2(t, x) m_\nu(t, x) dx &= \iint_{\Omega} 2(\nu - \nu') \Delta u_\nu (U m_\nu) + \nu \nabla m_\nu \nabla (U^2) + 2\nu' \Delta U (U m_\nu) \\ &\quad - (2(H(x, Du_\nu) - H(x, Du_{\nu'})) U m_\nu - D_p H(x, Du_\nu) D(U^2) m_\nu) \\ &\quad + 2(f(x, m_\nu) - f(x, m_{\nu'})) U m_\nu dx dt. \end{aligned} \quad (5.11)$$

Step 2. We estimate each term in the right-hand side of (5.11). In view of the last claim of Lemma 5.2, by Hölder's inequality and uniform boundedness of u, u_ν , we obtain

$$\iint_{\Omega} (\nu - \nu') \Delta u_\nu (U m_\nu) dx dt \leq (\nu - \nu') \|m_\nu^{1/2} \Delta u_\nu\|_{L^2(\Omega)} \|m_\nu^{1/2} U\|_{L^2(\Omega)} \leq C\nu.$$

By Proposition 3.3 and boundedness of U ,

$$\iint_{\Omega} \Delta U (U m_\nu) dx dt \leq \|\nabla U\|_{L^2(\Omega)}^2 \|m_\nu\|_\infty + \|\nabla U\|_{L^2(\Omega)} \|U\|_\infty \|m_\nu\|_\infty \leq C \|m_\nu\|_\infty.$$

Here $\|m_\nu\|_\infty$ is finite (depending on ν) because (u_ν, m_ν) is a classical solution. Similarly as done in the proof of Theorem 5.3, we have

$$\iint_{\Omega} \nabla m_\nu \nabla (U^2) dx dt \leq C \quad \text{and} \quad \iint_{\Omega} (f(x, m_\nu) - f(x, m_{\nu'})) U m_\nu dx dt \leq C\nu^{\frac{1}{2(1+\beta)}}$$

where C is uniform in ν, ν' .

Next, the condition (H8) and uniform boundedness of U again yield

$$\begin{aligned} &-\frac{1}{2} \iint_{\Omega} 2(H(x, Du_\nu) - H(x, Du)) U m_\nu - D_p H(x, Du_\nu) D(U^2) m_\nu dx dt \\ &= \iint_{\Omega} (H(x, Du) - H(x, Du_\nu) - D_p H(x, Du_\nu) (Du - Du_\nu)) U m_\nu dx dt \\ &\leq C \iint_{\Omega} |Du_\nu - Du|^2 m_\nu dx dt \leq C\nu^{\frac{1}{1+\beta}}, \end{aligned}$$

where in the last inequality, we applied Theorem 4.3 and Lemma 5.1.

Step 3. Putting the above estimates together into (5.11) yields

$$\int_{\mathbb{T}^d} U^2(t, x) m_\nu(t, x) dx dt \leq C\nu + C\nu^{\frac{1}{2(1+\beta)}} + C\|m_\nu\|_\infty \nu'.$$

Passing $\nu' \rightarrow 0$ yields the conclusion. \square

6. VANISHING VISCOSITY LIMIT – NONLOCAL COUPLING

In this section, we discuss mean field games with cost functions that are nonlocal and are regularizing on the set of probability measures. We allow the terminal data of u_ν to depend on $m_\nu(T, \cdot)$. Consider

$$\begin{cases} -\partial_t u_\nu - \nu \Delta u_\nu + H(x, Du_\nu) = f(x, m_\nu(t, \cdot)), \\ \partial_t m_\nu - \nu \Delta m_\nu - \nabla \cdot (m_\nu D_p H(x, Du_\nu)) = 0, \\ m_\nu(0, x) = \bar{m}(x), \quad u_\nu(T, x) = \bar{u}(x, m_\nu(T, \cdot)). \end{cases} \quad (6.1)$$

Let \mathcal{P} be the set of Borel probability measures μ on \mathbb{T}^d . The set \mathcal{P} can be endowed with the well-known Kantorovitch Rubinstein distance (or 1-Wasserstein distance), that is for any $\mu_1, \mu_2 \in \mathcal{P}$,

$$\mathbf{d}(\mu_1, \mu_2) := \sup_{\gamma \in \Pi(\mu_1, \mu_2)} \int_{\mathbb{T}^{2d}} |x - y| d\gamma(x, y)$$

where $\Pi(\mu_1, \mu_2)$ denotes all Borel probability measures on $\mathbb{T}^d \times \mathbb{T}^d$ that have μ_1 as its first marginal and μ_2 as its second marginal. We make the following assumptions (see [12, 43]). There exists $C > 0$ such that the following holds.

(H1') (Regularizing condition) $f : \mathbb{T}^d \times \mathcal{P} \rightarrow \mathbb{R}$ satisfies for any $\mu \in \mathcal{P}$,

$$\|f(\cdot, \mu)\|_{\mathcal{C}^2} \leq C,$$

and for any $x \in \mathbb{T}^d$ and $\mu_1, \mu_2 \in \mathcal{P}$,

$$|f(x, \mu_1) - f(x, \mu_2)| \leq C \mathbf{d}(\mu_1, \mu_2).$$

Moreover, f is strictly monotone in the second variable in the sense that for any $\mu_1, \mu_2 \in \mathcal{P}$, if $\mu_1 \neq \mu_2$, then

$$\int_{\mathbb{T}^d} (f(x, \mu_1) - f(x, \mu_2)) d(\mu_1 - \mu_2)(x) > 0.$$

If $\mu \in \mathcal{P}$ has a density function m , we write $f(x, m) := f(x, \mu)$.

(H2') (Conditions on the Hamiltonian) Assume (H2), and that $H = H(x, p)$ is \mathcal{C}^2 in both variables. Moreover, for all $x \in \mathbb{T}^d$ and $p \in \mathbb{R}^d$, we have

$$|H_x(x, p)| \leq C(1 + |p|^r), \quad |H_p(x, p)| \leq C(1 + |p|^{r-1}).$$

For any $R > 0$, there exists $C_R > 0$ such that for any $(x, p) \in \mathbb{T}^d \times \mathbb{R}^d$ with $|p| \leq R$,

$$|D_{xx}H(x, p)|, |D_{xp}H(x, p)| \leq C_R \quad \text{and} \quad 0 \leq D_{pp}H(x, p) \leq C_R I_d.$$

Moreover, H is convex in the second variable in the sense that there exists a non-negative function $c_1 : \mathbb{T}^d \rightarrow [0, \infty)$ such that for all $p, p' \in \mathbb{R}^d$ and $x \in \mathbb{T}^d$,

$$H(x, p) - H(x, p') - H_p(x, p)(p - p') \geq c_1(x)|p - p'|^2.$$

(H3') (Conditions on the initial and terminal data) $\bar{m} : \mathbb{T}^d \rightarrow \mathbb{R}$ is a \mathcal{C}^1 non-negative density function. $\bar{u} : \mathbb{T}^d \times \mathcal{P} \rightarrow \mathbb{R}$ satisfies for any $\mu \in \mathcal{P}$,

$$\|\bar{u}(\cdot, \mu)\|_{\mathcal{C}^2} \leq C,$$

and for any $x \in \mathbb{T}^d$ and $\mu_1, \mu_2 \in \mathcal{P}$,

$$|\bar{u}(x, \mu_1) - \bar{u}(x, \mu_2)| \leq C \mathbf{d}(\mu_1, \mu_2).$$

Moreover, we have the monotonicity condition: For any $m_1, m_2 \in \mathcal{P}$,

$$\int_{\mathbb{T}^d} (\bar{u}(x, m_1) - \bar{u}(x, m_2)) d(m_1 - m_2)(x) \geq 0.$$

If $\mu \in \mathcal{P}$ has a density function m , we write $\bar{u}(x, m) := \bar{u}(x, \mu)$.

Remark 6.1. 1. It is a classical result (see for example, [12, 43]) that under the assumptions of (H1')(H2')(H3'), there exists a unique classical solution to (6.1) when $\nu > 0$. When $\nu = 0$, (6.1) is still well-posed [12, 43], and the first equation in (6.1) is satisfied in the viscosity sense, and the second equation holds in the weak sense. It can be shown that $m_\nu(t, \cdot)$ is continuous in time with respect to Kantorovich-Rubinstein distance, and so $f(x, m)$ is continuous in time.

2. It was known that as $\nu \rightarrow 0$, the corresponding classical solutions (u_ν, m_ν) converge uniformly to the unique solution (u, m) of (6.1) with $\nu = 0$ (see [12]).

3. The typical example of f (of \bar{u}) is

$$f(x, m) = \int_{\mathbb{T}^d} g(y, (\phi * m)(y)) \phi(x - y) dy,$$

where $\phi : \mathbb{T}^d \rightarrow \mathbb{R}$ is a smooth even kernel and $g : \mathbb{T}^d \rightarrow \mathbb{R}$ is a smooth function such that g is strictly increasing in the second variable. Indeed, it is direct to see (see also [9, Example 4.1])

$$\begin{aligned} & \int_{\mathbb{T}^d} (f(x, m_1(\cdot)) - f(x, m_2(\cdot)))(m_1(x) - m_2(x)) dx \\ &= \int_{\mathbb{T}^d} (g(y, (\phi * m_1)(y)) - g(y, (\phi * m_2)(y)))(\phi * m_1(y) - \phi * m_2(y)) dy \geq 0. \end{aligned}$$

The following regularity results are consequences of (H1')(H2')(H3').

Lemma 6.1. There exists $C > 0$ such that for all $\nu \geq 0$ we have

$$m_\nu, |u_\nu|, |Du_\nu| \leq C \quad \text{in } \Omega, \quad (6.2)$$

$$\text{and} \quad D^2u_\nu \leq CI_d \quad \text{in } \Omega. \quad (6.3)$$

Proof. The comparison principle yields that u_ν is uniformly finite for all $\nu \geq 0$. The proof for Lipschitz regularity of u_ν follows from [1] (we also refer readers to [10] and [58]). Semiconcavity of u_ν is given in Theorem 5.3.6 [10] (as u_ν is uniformly Lipschitz continuous, by modifying $H(x, p)$ for large p one can assume that all second derivatives of H are uniformly finite). Finally, from the results in [12, Section 4.2], it follows that m_ν is uniformly bounded. \square

Now we prove the first main theorem of the section.

Theorem 6.2. Assume (H1')(H2')(H3') and let $c_1 = c_1(x)$ be from (H2'). For $\nu \in (0, 1]$, let (u_ν, m_ν) solve (6.1) and (u, m) solve (6.1) with $\nu = 0$. Then there exists $C > 0$ such that for all $\nu \in (0, 1]$ we have

$$\iint_{\Omega} (f(x, m_\nu) - f(x, m))(m_\nu - m) dx dt \leq C\nu^{1/2}, \quad (6.4)$$

$$\int_{\mathbb{T}^d} (\bar{u}(x, m_\nu(T, \cdot)) - \bar{u}(x, m(T, \cdot)))(m_\nu(T, \cdot) - m(T, \cdot)) dx \leq C\nu^{1/2}, \quad (6.5)$$

and

$$\iint_{\Omega} c_1(m_\nu + m) |Du_\nu - Du|^2 dx dt \leq C\nu^{1/2}. \quad (6.6)$$

Proof. The proof consists of four steps.

Step 1. We first show that the $W^{1,2}$ norm of $\nu^{1/2}m_\nu$ is uniformly bounded for all $\nu \in (0, 1]$. Note that (u_ν, m_ν) is a classical solution [12, 43]. So we can multiply the second equation in (6.1) by m_ν and then integrate over Ω to get

$$\int_{\mathbb{T}^2} m_\nu(T, x)^2 dx - \int_{\mathbb{T}^2} m_\nu(0, x)^2 dx + \nu \iint_{\Omega} |Dm_\nu|^2 dx dt = - \iint_{\Omega} m_\nu D_p H(x, Du_\nu) \cdot Dm_\nu dx dt.$$

Using (6.2) yields for some C independent of ν ,

$$\nu \iint_{\Omega} |Dm_\nu|^2 dx dt \leq \frac{1}{2} \iint_{\Omega} \nabla \cdot D_p H(x, Du_\nu) m_\nu^2 dx dt + C.$$

Note that $m_\nu, |Du_\nu| \leq C$ uniformly in ν by (6.2). Thus, it follows from (H2') and (6.3) that

$$\nu \iint_{\Omega} |Dm_\nu|^2 dx dt \leq C \iint_{\Omega} |D_{xp} H(x, Du_\nu)| + [\text{Tr}(D_{pp} H(x, Du_\nu) D^2 u_\nu)]_+ dx dt + C \leq C \quad (6.7)$$

for some constant C independent of $\nu \in (0, 1]$.

Step 2. Now we take any $\nu' \in (0, \nu)$, and define

$$U(t, x) := u_\nu(t, x) - u_{\nu'}(t, x), \quad M(t, x) := m_\nu(t, x) - m_{\nu'}(t, x).$$

It follows from the equations in (6.1) that

$$-\partial_t U - \nu \Delta u_\nu + \nu' \Delta u_{\nu'} + H(x, Du_\nu) - H(x, Du_{\nu'}) = f(x, m_\nu) - f(x, m_{\nu'}), \quad (6.8)$$

$$\partial_t M - \nu \Delta m_\nu + \nu' \Delta m_{\nu'} - \nabla \cdot (m_\nu D_p H(x, Du_\nu)) + \nabla \cdot (m_{\nu'} D_p H(x, Du_{\nu'})) = 0. \quad (6.9)$$

Let us multiply (6.8) by $-M$, and (6.9) by U , and integrate over Ω . We can do these as (u_ν, m_ν) and $(u_{\nu'}, m_{\nu'})$ are classical solutions. Since $M(0, \cdot) = 0$, we get

$$\begin{aligned} & \int_{\mathbb{T}^d} U(T, x) M(T, x) dx - \iint_{\Omega} \nu Du_\nu DM - \nu' Du_{\nu'} DM - \nu Dm_\nu DU + \nu' Dm_{\nu'} DU dx dt \\ &= \iint_{\Omega} (H(x, Du_\nu) - H(x, Du_{\nu'})) M + (m_\nu D_p H(x, Du_\nu) - m_{\nu'} D_p H(x, Du_{\nu'})) DU \\ & \quad - (f(x, m_\nu) - f(x, m_{\nu'})) M dx dt \\ &= - \iint_{\Omega} m_\nu (H(x, Du_{\nu'}) - H(x, Du_\nu) - D_p H(x, Du_\nu) (Du_{\nu'} - Du_\nu)) \\ & \quad + m_{\nu'} (H(x, Du_\nu) - H(x, Du_{\nu'}) - D_p H(x, Du_{\nu'}) (Du_\nu - Du_{\nu'})) \\ & \quad + (f(x, m_\nu) - f(x, m_{\nu'})) (m_\nu - m_{\nu'}) dx dt \\ &\leq - \iint_{\Omega} c_1(x) (m_\nu + m_{\nu'}) |Du_\nu - Du_{\nu'}|^2 + (f(x, m_\nu) - f(x, m_{\nu'})) (m_\nu - m_{\nu'}) dx dt, \end{aligned} \quad (6.10)$$

where we used (H2') in the inequality.

Step 3. Now we estimate the several terms in (6.10). By direct computations,

$$\begin{aligned} & \iint_{\Omega} \nu Du_\nu DM - \nu' Du_{\nu'} DM - \nu Dm_\nu DU + \nu' Dm_{\nu'} DU dx dt \\ &= \iint_{\Omega} (\nu - \nu') (Du_{\nu'} Dm_\nu - Du_\nu Dm_{\nu'}) dx dt = \iint_{\Omega} (\nu - \nu') (Du_{\nu'} Dm_\nu + \Delta u_\nu m_{\nu'}) dx dt. \end{aligned}$$

By (6.2) and (6.3), u_ν and $u_{\nu'}$ are uniformly Lipschitz continuous and semiconcave in space, and $m_{\nu'}$ is uniformly bounded. Combining these with $\nu' < \nu$ and (6.7), we obtain

$$\begin{aligned} & \iint_{\Omega} \nu Du_\nu DM - \nu' Du_{\nu'} DM - \nu Dm_\nu DU + \nu' Dm_{\nu'} DU \, dxdt \\ & \leq C\nu \iint_{\Omega} |Dm_\nu| + m_{\nu'} \, dxdt \leq C\nu \left(\iint_{\Omega} |Dm_\nu|^2 \, dxdt \right)^{1/2} + C\nu \leq C\nu^{1/2}. \end{aligned} \quad (6.11)$$

Thus (6.10) and (6.11) yield

$$\begin{aligned} & \int_{\mathbb{T}^d} U(T, x) M(T, x) \, dx + \iint_{\Omega} c_1(x) (m_\nu + m_{\nu'}) |Du_\nu - Du_{\nu'}|^2 \, dxdt \\ & \quad + \iint_{\Omega} (f(x, m_\nu) - f(x, m_{\nu'})) (m_\nu - m_{\nu'}) \, dxdt \leq C\nu^{1/2} \end{aligned}$$

In view of (H2')(H3'), the three terms on the left-hand side of the above are non-negative. Thus passing $\nu' \rightarrow 0$ yields (6.4) and (6.5).

Step 4. Finally, we pass $\nu' \rightarrow 0$ in

$$\iint_{\Omega} c_1(m_\nu + m_{\nu'}) |Du_\nu - Du_{\nu'}|^2 \, dxdt \leq C\nu^{1/2}.$$

Because $u_{\nu'}$ converges to u locally uniformly and $u_{\nu'}$ is uniformly semi-concave in x for all $\nu' \geq 0$, $Du_{\nu'}(t, \cdot)$ converges to $Du(t, \cdot)$ a.e. $x \in \mathbb{T}^d$. Thus passing ν' to 0 yields (6.6). \square

7. CONVERGENCE OF u_ν – NONLOCAL COUPLING

In this section, we proceed to show convergence results of u_ν as $\nu \rightarrow 0$. First, let us assume a strong condition (H4') on f and \bar{u} , and we prove pointwise convergence of u_ν to u . Later, we also consider a weaker condition (H4'').

(H4') There exists $C > 0$ such that for any $\varepsilon > 0$, if $\mu_1, \mu_2 \in \mathcal{P}$ satisfy

$$\int_{\mathbb{T}^d} (f(x, \mu_2) - f(x, \mu_1)) d(\mu_2 - \mu_1)(x) \leq \varepsilon,$$

then

$$\sup_{x \in \mathbb{T}^d} |f(x, \mu_2) - f(x, \mu_1)| \leq C\varepsilon^{1/2}.$$

And the same holds if we replace f by \bar{u} .

Remark 7.1. Note that if f (or \bar{u}) is of the form $f(x, m) = \int_{\mathbb{T}^d} g(y, (\phi * m)(y)) \phi(x - y) dy$, where ϕ is a smooth even function on \mathbb{T}^d and g is \mathcal{C}^1 on $\mathbb{T}^d \times \mathbb{R}$, then the condition reduces to

$$\text{for all } (x, z) \in \mathbb{T}^d \times \mathbb{R} \text{ we have } g_z(x, z) > 0.$$

Indeed suppose m_1, m_2 are two probability density function in \mathbb{T}^d . Since $|\phi * m_i| \leq \|\phi\|_\infty$, we have

$$|f(x, m_2) - f(x, m_1)| \leq \|\phi\|_\infty \int_{\mathbb{T}^d} |g(y, \phi * m_1) - g(y, \phi * m_2)| dy \leq C \|\phi * m_1 - \phi * m_2\|_{L^2(\mathbb{T}^d)}.$$

On the other hand, for $c := \inf_{(x, z) \in \mathbb{T}^d \times [-\|\phi\|_\infty, \|\phi\|_\infty]} g_z(x, z) > 0$, we get

$$\int_{\mathbb{T}^d} (f(x, m_2) - f(x, m_1)) (m_2 - m_1) \, dx \geq c \int_{\mathbb{T}^d} |\phi * m_1 - \phi * m_2|^2 \, dx.$$

So there exists $C > 0$ independent of m_1, m_2 such that for all $x \in \mathbb{T}^d$,

$$|f(x, m_2) - f(x, m_1)|^2 \leq C \int_{\mathbb{T}^d} (f(x, m_2) - f(x, m_1))(m_2 - m_1) dx.$$

We will need the following lemma.

Lemma 7.1. *Let $\varepsilon > 0$, and let H be as before, and $g : [0, T] \times \mathbb{T}^d \rightarrow \mathbb{R}$ be C^1 in space and Lipschitz continuous in time. Suppose that v and v_ε are Lipschitz continuous and satisfy*

$$\partial_t v + H(x, Dv) = g(t, x) \quad \text{and} \quad \partial_t v_\varepsilon - \varepsilon \Delta v_\varepsilon + H(x, Dv_\varepsilon) = g(t, x)$$

in the sense of viscosity, and $v(0, \cdot) = v_\varepsilon(0, \cdot)$ is a C^2 function. Then there exists $C > 0$ such that for all $\varepsilon > 0$,

$$|v(t, x) - v_\varepsilon(t, x)| \leq C\varepsilon^{1/2} \quad \text{in } [0, T] \times \mathbb{T}^d.$$

Though the proof is given by the classical viscosity solution approach (see e.g., [20] for the case when $H = H(p)$ and $g \equiv 0$), for readers' convenience, we provide it in the appendix.

Theorem 7.2. *Under the assumptions of Theorem 6.2, assume (H4'). Then there exists $C > 0$ such that for all $\nu \in (0, 1]$ we have*

$$\sup_{(t, x) \in [0, T] \times \mathbb{T}^d} |u(t, \cdot) - u_\nu(t, \cdot)| \leq C\nu^{1/4}.$$

Proof. For $\nu \in (0, 1]$, let w_ν be the unique solution to

$$-\partial_t w_\nu - \nu \Delta w_\nu + H(x, Dw_\nu) = f(x, m) \quad \text{with } w_\nu(T, x) = \bar{u}(x, m(T, \cdot)).$$

The condition (H4') and Theorem 6.2 imply that

$$|f(x, m) - f(x, m_\nu)|, \quad |w_\nu(T, x) - u_\nu(T, x)| = |\bar{u}(x, m(T, \cdot)) - \bar{u}(x, m_\mu(T, \cdot))| \leq C\nu^{1/4}.$$

Thus by comparing u_ν with $w_\nu \pm C\nu^{1/4}(T - t + 1)$ yields that

$$|u_\nu - w_\nu| \leq C\nu^{1/4} \quad \text{in } \Omega.$$

Note u satisfies

$$-\partial_t u + H(x, Du) = f(x, m) \quad \text{with } w(T, x) = \bar{u}(x, m(T, \cdot)),$$

in the sense of distribution, and u is also a viscosity solution by [37]. Moreover, by [12, Lemma 4.14] and (H1'), $f(x, m)$ is Lipschitz continuous in time. Then it follows from Lemma 7.1 that

$$\sup_{(t, x) \in \Omega} |w_\nu(t, x) - u(t, x)| \leq C\nu^{1/2},$$

which finishes the proof. □

Now we consider a weaker condition:

(H4'') There exists $C > 0$ such that for any $\varepsilon > 0$, if $\mu_1, \mu_2 \in \mathcal{P}$ satisfy

$$\int_{\mathbb{T}^d} (f(x, \mu_2) - f(x, \mu_1)) d(\mu_2 - \mu_1)(x) \leq \varepsilon,$$

then

$$\int_{\mathbb{T}^d} |f(x, \mu_2) - f(x, \mu_1)| dx \leq C\varepsilon^{1/2}.$$

And the same holds if we replace f by \bar{u} .

Since, under (H4''), it is only known that $f(x, m(t, \cdot))$ and $f(x, m_\nu(t, \cdot))$ are close in average from Theorem 6.2, we do not expect a pointwise strong convergence of u_ν to u . We will apply a dual equation method to prove an $L_t^\infty L_x^1$ convergence (see also [44]).

Theorem 7.3. *Under the assumptions of Theorem 6.2, assume (H4''). Then there exists $C > 0$ such that for all $\nu \in (0, 1]$ we have*

$$\sup_{t \in [0, T]} \|u(t, \cdot) - u_\nu(t, \cdot)\|_{L^1(\mathbb{T}^d)} \leq C\nu^{1/4}.$$

Proof. For $\nu \in (0, 1]$, let w_ν be the unique solution to

$$-\partial_t w_\nu - \nu \Delta w_\nu + H(x, Dw_\nu) = f(x, m) \quad \text{with } w_\nu(T, x) = \bar{u}(x, m(T, \cdot)).$$

By Lemma 6.1, there exists $C > 0$ such that for all $\nu \in (0, 1]$,

$$|Du|, |Du_\nu|, |w_\nu| \leq C \quad \text{and} \quad D^2 u_\nu, D^2 w \leq CI_d \quad \text{in } \Omega. \quad (7.1)$$

We will compare u_ν with w_ν and then w_ν with u .

Step 1. Consider $W := w_\nu - u_\nu$, which then satisfies

$$-\partial_t W - \nu \Delta W + G \cdot DW = f(x, m) - f(x, m_\nu) \quad (7.2)$$

with

$$W(T, x) = w_\nu(T, x) - u_\nu(T, x) = \bar{u}(x, m(T, \cdot)) - \bar{u}(x, m_\nu(T, \cdot)),$$

and

$$G(t, x) := \int_0^1 D_p H(x, sDw_\nu + (1-s)Du_\nu) ds.$$

Recall that

$$-\partial_t u_\nu - \nu \Delta u_\nu + H(x, Du_\nu) = f(x, m_\nu) \quad \text{with } u_\nu(T, x) = \bar{u}(x, m_\nu(T, \cdot)).$$

By (H4'') and Theorem 6.2,

$$\begin{aligned} \iint_\Omega |f(x, m) - f(x, m_\nu)| dx dt &\leq \int_0^t C \left(\int_{\mathbb{T}^d} (f(x, m) - f(x, m_\nu))(m - m_\nu) dx \right)^{1/2} dt \\ &\leq C \left(\iint_\Omega (f(x, m) - f(x, m_\nu))(m - m_\nu) dx dt \right)^{1/2} \leq C\nu^{1/4}. \end{aligned} \quad (7.3)$$

Similarly, by (6.5) of Theorem 6.2,

$$\int_{\mathbb{T}^d} |\bar{u}(x, m(T, \cdot)) - \bar{u}(x, m_\nu(T, \cdot))| dx \leq C\nu^{1/4}. \quad (7.4)$$

Step 2. For any fixed $t_1 \in [0, T]$, we consider the dual equation of (7.2) in $[t_1, T] \times \mathbb{T}^d$:

$$\psi_t - \nu \Delta \psi - \nabla \cdot (G\psi) = 0 \quad \text{with } \psi(t_1, \cdot) = \psi_0,$$

where ψ_0 is a smooth function on \mathbb{T}^d . By Divergence Theorem, we have

$$\frac{d}{dt} \int_{\mathbb{T}^d} W\psi dx = - \int_{\mathbb{T}^d} (f(x, m) - f(x, m_\nu))\psi dx.$$

In view of (7.3) and (7.4), integrating the above equality in the time interval $[t_1, T]$ yields

$$\begin{aligned} \left| \int_{\mathbb{T}^d} W(t_1, x) \psi_0(x) dx \right| &\leq \left| \int_{\mathbb{T}^d} W(T, x) \psi(T, x) dx \right| + C\nu^{1/4} \|\psi\|_{L^\infty([t_1, T] \times \mathbb{T}^d)} \\ &\leq C\nu^{1/4} \|\psi\|_{L^\infty([t_1, T] \times \mathbb{T}^d)}. \end{aligned} \quad (7.5)$$

Step 3. Now, we estimate $\|\psi\|_{L^\infty([t_1, T] \times \mathbb{T}^d)}$ in terms of $\|\psi_0\|_{L^\infty(\mathbb{T}^d)}$. It follows from the equation of ψ that for any $n \geq 2$ an even number,

$$\frac{d}{dt} \int_{\mathbb{T}^d} \psi^n dx + n(n-1)\nu \int_{\mathbb{T}^d} |D\psi|^2 \psi^{n-2} dx = -n(n-1) \int_{\mathbb{T}^d} (G \cdot D\psi) \psi^{n-1}.$$

So for any $t \in [t_1, T]$,

$$\int_{\mathbb{T}^d} \psi(t, x)^n dx \leq \|\psi_0\|_\infty^n + (n-1) \int_{t_1}^t \int_{\mathbb{T}^d} (\nabla \cdot G(s, x)) \psi^n(s, x) dx ds.$$

This yields that if $\nabla \cdot G(s, x)$ is uniformly bounded from above for all ν , then by Gronwall's inequality,

$$\|\psi(t, \cdot)\|_{L^n(\mathbb{T}^d)} \leq \|\psi_0\|_{L^\infty(\mathbb{T}^d)} \exp\left(\|(\nabla \cdot G)_+\|_{L^\infty([t_1, T] \times \mathbb{T}^d)}(t - t_1)\right).$$

Passing $n \rightarrow \infty$ yields

$$\|\psi\|_{L^\infty([t_1, T] \times \mathbb{T}^d)} \leq \|\psi_0\|_{L^\infty(\mathbb{T}^d)} \exp\left(\|(\nabla \cdot G)_+\|_{L^\infty([t_1, T] \times \mathbb{T}^d)} T\right).$$

Applying this in (7.5), we get for some $C > 0$,

$$\left| \int_{\mathbb{T}^d} W(t_1, x) \psi_0(x) dx \right| \leq C\nu^{1/4} \|\psi_0\|_{L^\infty(\mathbb{T}^d)}$$

holds for all smooth ψ_0 , which implies that

$$\|w_\nu(t_1, \cdot) - u_\nu(t_1, \cdot)\|_{L^1(\mathbb{T}^d)} \leq C\nu^{1/4} \quad \text{for any } t_1 \in [0, T]. \quad (7.6)$$

Step 4. To finish the proof of (7.6), we now show that $\nabla \cdot G(t, x)$ is uniformly bounded from above. Indeed we have

$$\nabla \cdot G = \int_0^1 \sum_i \frac{\partial^2 H}{\partial p_i \partial x_i} + \left(s \sum_{i,j} \frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial^2 w_\nu}{\partial x_i \partial x_j} + (1-s) \sum_{i,j} \frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial^2 u_\nu}{\partial x_i \partial x_j} \right) ds.$$

By (7.1) and (H2'), there exists $C > 0$ such that for all $\nu, s \in [0, 1]$,

$$\frac{\partial^2 H}{\partial p_i \partial x_i}(x, sDw_\nu + (1-s)Du_\nu) \leq C,$$

$$\frac{\partial^2 H}{\partial p_i \partial p_j}, \quad \frac{\partial^2 w_\nu}{\partial x_i \partial x_j}, \quad \frac{\partial^2 u_\nu}{\partial x_i \partial x_j} \leq C.$$

Therefore, we obtain that $\nabla \cdot G \leq C$ for some C independent of ν .

Step 5. Due to (7.6), to conclude the theorem, it suffices to show that

$$\sup_{(t,x) \in \Omega} |w_\nu(t, x) - u(t, x)| \leq C\nu^{1/2}.$$

This is a consequence of Lemma 7.1 (see also the proof of Theorem 7.2). \square

8. OPEN PROBLEMS

In this section, we collect a few open problems related to the convergence of vanishing viscosity approximations for MFGs and its connection to the KPZ equation.

- (1) In Theorem 4.3, we proved a rate of $\nu^{\frac{1}{2(1+\beta)}}$ for the convergence of (m_ν, Du_ν) in some Sobolev norm assuming that $H(\cdot, p)$ grows as $|p|^r$, and $f(\cdot, m)$ grows as m^{r-1} . We know that if $\frac{1}{q} + \frac{1}{r} \leq 1$, this rate is $\nu^{\frac{1}{4}}$ which is independent of the dimension d ; while $\frac{1}{q} + \frac{1}{r} > 1$, the exponent $\frac{1}{2(1+\beta)} \asymp d^{-1}$. Is the rate $\nu^{\frac{1}{2(1+\beta)}}$ is tight, or does the threshold $\frac{1}{q} + \frac{1}{r} = 1$ induces a phase transition in the dimension dependence of the rate exponent of vanishing viscosity approximations for MFGs?
- (2) In Theorems 4.3, 5.3 and 5.4, we proved the convergence (rate) of (m_ν, u_ν) in L^2 norm. For the (1+1)-dimensional KPZ equation, it is known that under narrow wedge initial condition, u_ν converges locally uniformly but without rate.
 - (a) Does the convergence still hold in a stronger sense, e.g. locally uniformly for u_ν , and do we get the same convergence rate?
 - (b) Can we relax the condition $q = 2$, (5.6) and $r = 2$ in these theorems?

In Theorem 5.4, we proved the convergence of u_ν in L^2 norm weighted by m_ν . Can we prove the same rate in L^2 norm weighted by m (and further locally uniformly)?

- (3) For the (1+1)-dimensional KPZ equation, Theorem 4.3 implies that ρ_ν converges in L^2 norm with a rate $\nu^{\frac{1}{4}}$; Theorem 5.4 shows that h_ν converges in a weighted L^2 norm with rate $\nu^{\frac{1}{8}}$. This relies on the Cole-Hopf transform from the SHE to the KPZ equation, which underlies the weak noise theory. Is there a higher dimensional weak noise theory to connect large deviations of the KPZ equation to vanishing viscosity for MFGs?

We hope that our convergence analysis of vanishing viscosity approximations for MFGs may trigger further research connecting the KPZ equation and MFGs in dimension $d \geq 2$, and on the dimension effect as $d \rightarrow \infty$.

Acknowledgments: We thank Sharon Xuan Di for various pointers to the applications of first-order MFGs in transportation, and Yier Lin for telling us about the weak noise theory for the KPZ equation. We thank Andrzej Świąch for remarks which lead to the first part of Section 7. We also thank François Delarue, Dan Lacker and Jianfeng Zhang for helpful discussions. The work of W. Tang is supported by NSF grants DMS-2113779 and DMS-2206038, and a start-up grant at Columbia University.

APPENDIX A.

Lemma A.1. *Suppose $q, q' > 1$ satisfying $\frac{1}{q} + \frac{1}{q'} = 1$, and $\sigma > 0$. Then for $c_0 := (\max\{q, q'\})^{-1}$, we have for all $m, \alpha \geq 0$,*

$$\frac{1}{q}(\sigma m)^q + \frac{1}{q'}(\alpha/\sigma)^{q'} \geq m\alpha + c_0((\sigma m)^{q/2} - (\alpha/\sigma)^{q'/2})^2.$$

Proof. By replacing σm by m , and α/σ by α , we can assume without loss of generality that $\sigma = 1$. Also we can assume that $m, \alpha > 0$ and $q > 2 > q' > 1$, otherwise the proof is direct. In

this case, $c_0 = \frac{1}{q}$. By direct computations, the inequality is reduced to

$$\left(\frac{1}{q'} - \frac{1}{q}\right)\alpha^{q'-1} + \frac{2}{q}m^{q/2}\alpha^{q'/2-1} \geq m. \quad (\text{A.1})$$

By Hölder's inequality,

$$\gamma_1\alpha^{q'-1} + \gamma_2m^{q/2}\alpha^{q'/2-1} \geq \alpha^{\gamma_1(q'-1)+\gamma_2(q'/2-1)}m^{\gamma_2(q/2)} = m,$$

where $\gamma_1 := 1 - 2/q, \gamma_2 := 2/q$. This implies (A.1) immediately, which finishes the proof.

It is also not hard to see from the proof that c_0 can not be larger than $\frac{1}{q}$. Otherwise, if $c_0 > \frac{1}{q}$, we need

$$\left(\frac{1}{q'} - c_0\right)\alpha^{q'-1} + \frac{2}{q}m^{q/2}\alpha^{q'/2-1} \geq m + \left(c_0 - \frac{1}{q}\right)m^q/\alpha,$$

but this cannot be true as passing $\alpha \rightarrow 0$ leads to a contradiction. \square

APPENDIX B. PROOF OF LEMMA 7.1

Let us only prove that $\sup_{\Omega}(v - v_{\varepsilon})$ is small. The other side of the estimate follows the same. We split the proof into five steps.

Step 1. First of all, since v, v_{ε} are uniformly Lipschitz continuous, by modifying $H(x, p)$ in the region where $|p|$ is large (see e.g., section 5 of [56]), we can further assume without loss of generality that for some $C > 0$,

$$|H_x(x, p)| \leq C \quad \text{for all } (x, p) \in \mathbb{T}^d \times \mathbb{R}^d. \quad (\text{B.1})$$

Let us take $T \geq 1$. Suppose $(t_0, x_0) \in [0, T] \times \mathbb{T}^d$ is such that

$$3\sigma := v(t_0, x_0) - v_{\varepsilon}(t_0, x_0) = \sup_{(t, x) \in \Omega} [v(t, x) - v_{\varepsilon}(t, x)] > 0. \quad (\text{B.2})$$

Below we will show $\sigma \leq C\sqrt{\varepsilon}$ where C depends on T , the Lipschitz constant of v and v_{ε} , and the assumptions.

Step 2. Consider a smooth function $\varphi : \mathbb{R}^{d+1} \rightarrow [0, 1]$ such that

- (i) $\varphi(t, x) = 1 - t^2 - |x|^2$ if $t^2 + |x|^2 < 1/2$,
- (ii) $0 \leq \varphi(t, x) \leq 1/2$ if $t^2 + |x|^2 > 1/2$, and $\varphi(t, x) = 0$ if $t^2 + |x|^2 > 1$.

For $\delta > 0$, denote $\varphi_{\delta}(t, x) := \varphi(t/\delta, x/\delta)$, $\Omega := (0, T) \times \mathbb{T}^d$, and

$$L := \sup\{v(t, x), -v_{\varepsilon}(t, x) : (t, x) \in \Omega\} + 1 \geq 1,$$

Next, we define $\Phi_{\varepsilon, \delta} : [0, T]^2 \times \mathbb{T}^{2d} \rightarrow \mathbb{R}$ by

$$\Phi_{\varepsilon, \delta}(t, s, x, y) := v(t, x) - v_{\varepsilon}(s, y) - \sigma(2T - t - s)/T + 8L\varphi_{\delta}(t - s, x - y).$$

There exists $(t_1, s_1, x_1, y_1) \in [0, T]^2 \times \mathbb{T}^{2d}$ such that

$$\Phi_{\varepsilon, \delta}(t_1, s_1, x_1, y_1) = \max_{[0, T]^2 \times \mathbb{T}^{2d}} \Phi_{\varepsilon, \delta}(t, s, x, y), \quad (\text{B.3})$$

and by (B.2) and $\varphi_{\delta}(0, 0) = 1$,

$$\Phi_{\varepsilon, \delta}(t_1, s_1, x_1, y_1) \geq \Phi_{\varepsilon, \delta}(t_0, t_0, x_0, x_0) \geq 8L + \sigma. \quad (\text{B.4})$$

Since $\max\{v(t_1, x_1), -v_{\varepsilon}(s_1, y_1)\} \leq L$,

$$\Phi_{\varepsilon, \delta}(t_1, s_1, x_1, y_1) \leq 2L + 8L\varphi_{\delta}(t_1 - s_1, x_1 - y_1),$$

which, together with (B.4), implies $\varphi_\delta(t_1 - s_1, x_1 - y_1) \geq 3/4$. Then by (i), we get

$$\varphi_\delta(t - s_1, x - y_1) = 1 - (|t - s_1|^2 + |x - y_1|^2)/\delta^2, \quad (\text{B.5})$$

whenever $|t - t_1|, |x - x_1| \leq \delta/C$ for some $C > 0$.

Step 3. Now, in view of (B.3), the mapping

$$(t, x) \mapsto v(t, x) + \sigma t/T + 8L\varphi_\delta(t - s_1, x - y_1). \quad (\text{B.6})$$

is maximized at $(t, x) = (t_1, x_1)$. As v is uniformly Lipschitz continuous by the assumption, we get that

$$\begin{aligned} |D\varphi_\delta(t_1 - s_1, x_1 - y_1)| &\leq C/L \quad \text{and} \\ |\partial_t \varphi_\delta(t_1 - s_1, x_1 - y_1)| &\leq C(1 + \sigma/T)/L. \end{aligned}$$

By (B.5) and $\sigma \leq L$, these yield

$$|x_1 - y_1| \leq C\delta^2/L \quad \text{and} \quad (\text{B.7})$$

$$|t_1 - s_1| \leq C\delta^2(1 + \sigma/T)/L. \quad (\text{B.8})$$

Step 4. We firstly assume that $t_1, s_1 > 0$. In view of (B.6), the viscosity solution test for v yields

$$-\sigma/T - 8L \partial_t \varphi_\delta(t_1 - s_1, x_1 - y_1) + H(x_1, -8L D\varphi_\delta(t_1 - s_1, x_1 - y_1)) \geq g(t_1, x_1).$$

Similarly, because

$$(s, y) \rightarrow v_\varepsilon(s, y) - \frac{\sigma}{T}s - 8L\varphi_\delta(t_1 - s, x_1 - y)$$

is minimized at (s_1, y_1) , the viscosity solution test yields

$$\begin{aligned} \sigma/T - 8L \partial_t \varphi_\delta(t_1 - s_1, x_1 - y_1) + H(y_1, -8L D\varphi_\delta(t_1 - s_1, x_1 - y_1)) \\ - 8\varepsilon L \Delta \varphi_\delta(t_1 - s_1, x_1 - y_1) \leq g(s_1, y_1). \end{aligned}$$

Thus we get

$$\begin{aligned} 2\sigma/T \leq H(x_1, -8L D\varphi_\delta(t_1 - s_1, x_1 - y_1)) - H(y_1, -8L D\varphi_\delta(t_1 - s_1, x_1 - y_1)) \\ + 8\varepsilon L \Delta \varphi_\delta(t_1 - s_1, x_1 - y_1) + g(s_1, y_1) - g(t_1, x_1). \end{aligned} \quad (\text{B.9})$$

For the second order term, the definition of φ_δ implies

$$8\varepsilon L \Delta \varphi_\delta(t_1 - s_1, x_1 - y_1) \leq C\varepsilon L \delta^{-2}. \quad (\text{B.10})$$

Using (B.9), (B.10), (B.1) and Lipschitz continuity of g yields for some universal C ,

$$2\sigma/T \leq C\varepsilon L \delta^{-2} + C(|x_1 - y_1| + |t_1 - s_1|) \leq C\varepsilon L \delta^{-2} + C\delta^2(1 + \sigma/T)/L$$

where in the last inequality we used (B.7) and (B.8). Finally, taking $\delta := L^{1/2}\varepsilon^{1/4}$ yields $\sigma \leq C\sqrt{\varepsilon}$ when ε is sufficiently small depending only on T, C . This finishes the proof of the upper bound of $\sup_\Omega(v - v_\varepsilon)$ in the case when $t_1, s_1 > 0$.

Step 5. Finally, suppose that one of t_1 and s_1 equals to T . Let us only prove for the case when $t_1 = 0$. By (B.4),

$$8L + \sigma \leq \Phi_{\varepsilon, \delta}(t_1, s_1, x_1, y_1) \leq v(t_1, x_1) - v_\varepsilon(s_1, y_1) + 8L\varphi_\delta(t_1 - s_1, x_1 - y_1).$$

Again using that v_ε is uniformly Lipschitz continuous, this, (B.7) and (B.8) yield

$$\begin{aligned} 8L + \sigma \leq |v(0, x_1) - v(0, y_1)| + |v_\varepsilon(0, y_1) - v_\varepsilon(s_1, y_1)| + 8L\varphi_\delta(-s_1, x_1 - y_1) \\ \leq C(|x_1 - y_1| + |s_1|) + 8L \leq C\delta^2(1 + \sigma/T)/L + 8L. \end{aligned}$$

Since $\delta = L^{1/2}\varepsilon^{1/4}$, this yields $\sigma \leq C\sqrt{\varepsilon}$ for some universal $C > 0$ when ε is small.

REFERENCES

- [1] S. N. Armstrong and H. V. Tran. Viscosity solutions of general viscous Hamilton-Jacobi equations. *Math. Ann.*, 361(3-4):647–687, 2015.
- [2] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math. (2)*, 161(1):223–342, 2005.
- [3] Y. Brenier. Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations. *Comm. Pure Appl. Math.*, 52(4):411–452, 1999.
- [4] A. Bressan, F. Huang, Y. Wang, and T. Yang. On the convergence rate of vanishing viscosity approximations for nonlinear hyperbolic systems. *SIAM J. Math. Anal.*, 44(5):3537–3563, 2012.
- [5] A. Bressan and T. Yang. On the convergence rate of vanishing viscosity approximations. *Comm. Pure Appl. Math.*, 57(8):1075–1109, 2004.
- [6] S. Cacace, F. Camilli, and A. Goffi. A policy iteration method for mean field games. *ESAIM Control Optim. Calc. Var.*, 27:Paper No. 85, 19, 2021.
- [7] F. Camilli. A policy iteration method for mean field games. *IFAC-PapersOnLine*, 55(30):406–411, 2022.
- [8] F. Camilli and Q. Tang. Rates of convergence for the policy iteration method for mean field games systems. *J. Math. Anal. Appl.*, 512(1):Paper No. 126138, 18, 2022.
- [9] P. Cannarsa and R. Capuani. Existence and uniqueness for mean field games with state constraints. *PDE models for multi-agent phenomena*, pages 49–71, 2018.
- [10] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, volume 58 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2004.
- [11] F. Caravenna, R. Sun, and N. Zygouras. Universality in marginally relevant disordered systems. *Ann. Appl. Probab.*, 27(5):3050–3112, 2017.
- [12] P. Cardaliaguet. Notes on mean field games. 2013. Available at <https://www.ceremade.dauphine.fr/~cardaliaguet/MFG20130420.pdf>.
- [13] P. Cardaliaguet. Weak solutions for first order mean field games with local coupling. In *Analysis and geometry in control theory and its applications*, volume 11 of *Springer INdAM Ser.*, pages 111–158. Springer, Cham, 2015.
- [14] P. Cardaliaguet and P. J. Graber. Mean field games systems of first order. *ESAIM Control Optim. Calc. Var.*, 21(3):690–722, 2015.
- [15] P. Cardaliaguet, P. J. Graber, A. Porretta, and D. Tonon. Second order mean field games with degenerate diffusion and local coupling. *NoDEA Nonlinear Differential Equations Appl.*, 22(5):1287–1317, 2015.
- [16] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, and A. Porretta. Long time average of mean field games. *Netw. Heterog. Media*, 7(2):279–301, 2012.
- [17] R. Carmona. Applications of mean field games in financial engineering and economic theory. 2020. arXiv:2012.05237.
- [18] S. Chatterjee and A. Dunlap. Constructing a solution of the (2+1)-dimensional KPZ equation. *Ann. Probab.*, 48(2):1014–1055, 2020.
- [19] I. Corwin. The Kardar-Parisi-Zhang equation and universality class. *Random Matrices Theory Appl.*, 1(1):1130001, 76, 2012.
- [20] M. G. Crandall and P.-L. Lions. Two approximations of solutions of Hamilton-Jacobi equations. *Math. Comp.*, 43(167):1–19, 1984.
- [21] E. DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.
- [22] B. Djehiche, A. Tcheukam, and H. Tembine. Mean-field-type games in engineering. *AIMS Electronics and Electrical Engineering*, 1(1):18–73, 2017.
- [23] L. C. Evans. Adjoint and compensated compactness methods for Hamilton-Jacobi PDE. *Arch. Ration. Mech. Anal.*, 197(3):1053–1088, 2010.
- [24] R. Ferreira, D. Gomes, and T. Tada. Existence of weak solutions to time-dependent mean-field games. *Nonlinear Anal.*, 212:Paper No. 112470, 31, 2021.
- [25] R. C. Fetecau, H. Huang, D. Messenger, and W. Sun. Zero-diffusion limit for aggregation equations over bounded domains. *Discrete Contin. Dyn. Syst.*, 42(10):4905–4936, 2022.

- [26] R. C. Fetecau, M. Kovacic, and I. Topaloglu. Swarming in domains with boundaries: approximation and regularization by nonlinear diffusion. *Discrete Contin. Dyn. Syst. Ser. B*, 24(4):1815–1842, 2019.
- [27] P. Gaudreau Lamarre, Y. Lin, and L.-C. Tsai. KPZ equation with a small noise, deep upper tail and limit shape. 2021. arXiv:2106.13313.
- [28] D. A. Gomes, E. Pimentel, and H. Sánchez-Morgado. Time-dependent mean-field games in the superquadratic case. *ESAIM Control Optim. Calc. Var.*, 22(2):562–580, 2016.
- [29] D. A. Gomes, E. A. Pimentel, and H. Sánchez-Morgado. Time-dependent mean-field games in the subquadratic case. *Comm. Partial Differential Equations*, 40(1):40–76, 2015.
- [30] P. J. Graber. Optimal control of first-order Hamilton-Jacobi equations with linearly bounded Hamiltonian. *Appl. Math. Optim.*, 70(2):185–224, 2014.
- [31] P. J. Graber and A. R. Mészáros. Sobolev regularity for first order mean field games. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 35(6):1557–1576, 2018.
- [32] P. J. Graber, A. R. Mészáros, F. J. Silva, and D. Tonon. The planning problem in mean field games as regularized mass transport. *Calc. Var. Partial Differential Equations*, 58(3):Paper No. 115, 28, 2019.
- [33] K. Huang, X. Di, Q. Du, and X. Chen. A game-theoretic framework for autonomous vehicles velocity control: bridging microscopic differential games and macroscopic mean field games. *Discrete Contin. Dyn. Syst. Ser. B*, 25(12):4869–4903, 2020.
- [34] K. Huang, X. Di, Q. Du, and X. Chen. Scalable traffic stability analysis in mixed-autonomy using continuum models. *Transp. Res. C: Emerg. Technol.*, 111:616–630, 2020.
- [35] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ϵ -Nash equilibria. *IEEE Trans. Automat. Control*, 52(9):1560–1571, 2007.
- [36] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, 6(3):221–251, 2006.
- [37] H. Ishii. On the equivalence of two notions of weak solutions, viscosity solutions and distribution solutions. *Funkcial. Ekvac*, 38(1):101–120, 1995.
- [38] B. Jovanovic and R. W. Rosenthal. Anonymous sequential games. *J. Math. Econom.*, 17(1):77–87, 1988.
- [39] P. Kachroo, S. Agarwal, and S. Sastry. Inverse problem for non-viscous mean field control: example from traffic. *IEEE Trans. Automat. Control*, 61(11):3412–3421, 2016.
- [40] A. Kamenev, B. Meerson, and P. V. Sasorov. Short-time height distribution in the one-dimensional Kardar-Parisi-Zhang equation: starting from a parabola. *Phys. Rev. E*, 94(3):032108, 9, 2016.
- [41] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.
- [42] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.
- [43] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.
- [44] C.-T. Lin and E. Tadmor. L^1 -stability and error estimates for approximate Hamilton-Jacobi solutions. *Numer. Math.*, 87(4):701–735, 2001.
- [45] Y. Lin and L.-C. Tsai. Short time large deviations of the KPZ equation. *Comm. Math. Phys.*, 386(1):359–393, 2021.
- [46] Y. Lin and L.-C. Tsai. A lower-tail limit in the weak noise theory. 2022. arXiv:2210.05629.
- [47] J. Magnen and J. Unterberger. The scaling limit of the KPZ equation in space dimension 3 and higher. *J. Stat. Phys.*, 171(4):543–598, 2018.
- [48] B. Meerson, E. Katzav, and A. Vilenkin. Large deviations of surface height in the Kardar-Parisi-Zhang equation. *Phys. Rev. Lett.*, 116(7):070601, 5, 2016.
- [49] S. Muñoz. Classical and weak solutions to local first-order mean field games through elliptic regularity. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 39(1):1–39, 2022.
- [50] C. Orrieri, A. Porretta, and G. Savaré. A variational approach to the mean field planning problem. *J. Funct. Anal.*, 277(6):1868–1957, 2019.
- [51] A. Porretta. On the planning problem for the mean field games system. *Dyn. Games Appl.*, 4(2):231–256, 2014.
- [52] A. Porretta. Weak solutions to Fokker-Planck equations and mean field games. *Arch. Ration. Mech. Anal.*, 216(1):1–62, 2015.

- [53] A. Porretta. Regularizing effects of the entropy functional in optimal transport and planning problems. *J. Funct. Anal.*, 284(3):Paper No. 109759, 65, 2023.
- [54] A. Proskinski and F. Santambrogio. Global-in-time regularity via duality for congestion-penalized mean field games. *Stochastics*, 89(6-7):923–942, 2017.
- [55] J. Quastel. Introduction to KPZ. In *Current Developments in Mathematics, 2011*, pages 125–194. Int. Press, Somerville, MA, 2012.
- [56] W. Tang, H. V. Tran, and Y. P. Zhang. Policy iteration for the deterministic control problems—a viscosity approach. 2023. arXiv:2301.00419.
- [57] H. V. Tran. Adjoint methods for static Hamilton-Jacobi equations. *Calc. Var. Partial Differential Equations*, 41(3-4):301–319, 2011.
- [58] H. V. Tran. *Hamilton-Jacobi equations—theory and applications*, volume 213 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2021.
- [59] L.-C. Tsai. Integrability in the weak noise theory. 2022. arXiv:2204.00614.
- [60] Y. Zhang. On continuity equations in space-time domains. *Discrete Contin. Dyn. Syst.*, 38(10):4837–4873, 2018.

(W. Tang) DEPARTMENT OF INDUSTRIAL ENGINEERING AND OPERATIONS RESEARCH, COLUMBIA UNIVERSITY, S.W. MUDD BUILDING, 500 W 120TH ST, NEW YORK, NY 10027

Email address: wt2319@columbia.edu

(Y. P. Zhang) DEPARTMENT OF MATHEMATICS AND STATISTICS, AUBURN UNIVERSITY, PARKER HALL, 221 ROOSEVELT CONCOURSE, AUBURN, AL 36849

Email address: yzhangpaul@auburn.edu