Optimal resource allocation for maintaining system solvency

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Abstract. We study an optimal allocation problem for a system of independent Brownian agents whose states evolve under a limited shared control. At each time, a unit of resource can be divided and allocated across components to increase their drifts, with the objective of maximizing either (i) the probability that all components avoid ruin, or (ii) the expected number of components that avoid ruin. We derive the associated Hamilton–Jacobi–Bellman equations on the positive orthant with mixed boundary conditions at the absorbing boundary and at infinity, and we identify drift thresholds separating trivial and nontrivial regimes. For the all-survive criterion, we establish existence, uniqueness, and smoothness of a bounded classical solution and a verification theorem linking the PDE to the stochastic control value function. We then investigate a conjecture on the optimality of a socialistic allocation rule: in the nonnegative-drift regime, we prove that it is optimal for the all-survive value function, while it is not optimal for the count-survivors criterion, by exhibiting a two-dimensional counterexample and then lifting it to all dimensions.

Key words: stochastic control, allocation problem, HJB equation, socialistic policy

1 Introduction

We study a stochastic allocation problem for a system of n independent Brownian agents whose states evolve under a shared control budget. An agent is annihilated when her state level hits zero, while the planner can distribute one unit of resource per unit time across the agents to increase their state levels. The objective is to allocate the resource dynamically so as to maximize long-run survival. Two natural criteria lead to qualitatively different control problems:

(1) maximize the probability that all agents survive forever;
(2) maximize the expected number of agents that survive forever.

These two objectives capture, respectively, a system-wide robustness criterion and an aggregate survival criterion.

Many real-world allocation problems share such a common tension: a planner can distribute a limited resource among several units whose situations evolve randomly, and the planner's objective may reflect either efficiency, meaning maximizing the number of units that do well, or robustness, meaning avoiding catastrophic failure of the system as a whole. A vivid example comes from emergency departments: hospitals allocate scarce resources such as physician time, ICU beds, ventilators, and diagnostic capacity to a finite number of patients, each with uncertain health trajectory. Operations research models [1, 15] emphasize that decisions must be made in real time under uncertainty, and that different prioritization rules can strongly affect system performance and patient outcomes. From a normative viewpoint, triage guidelines balance several ethical principles [27], such as "save the most lives", "protect the worst-off", and "treat people equally", and these principles may conflict in scarce-resource settings [24].

A second viewpoint comes from social welfare design: should a planner concentrate effort on those closest to failure, following a maximin or Rawlsian intuition [23], or instead allocate effort where it yields the greatest aggregate benefit, following a more utilitarian logic? This question is closely related to an interpretation proposed by McKean and Shepp [21], who analyzed competing allocation principles within a stochastic framework for distributing resources across (two) companies. They compared two distinct government tax policies for corporations: the republican policy gives tax breaks to the richer companies, while the democratic policy gives tax breaks to the weaker companies in the hope of keeping them alive and thereby reducing unemployment.

A central (and natural) question is:

Does allocating resources to the weakest agent/entity yield the highest marginal benefit?

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Such an allocation rule is referred to as the *push-the-laggard strategy* [26]. In the emergency department analogy, this resembles a sickest-first rule; in a social welfare analogy, it resembles a prioritize-the-least-advantaged rule. Our main goal is to show that the answer to the above question depends sharply on the objective: prioritizing the weakest is optimal for a stringent system-survival criterion, but may fail for a maximize-the-number-of-survivors criterion, extending [21] for any finite number of agents/entities.

Technically, we analyze this problem through its Hamilton–Jacobi–Bellman (HJB) equation on the positive orthant, supplemented with absorbing boundary conditions at zero and recursive asymptotic conditions at infinity. Our first result identifies drift thresholds distinguishing between trivial and nontrivial regimes. In the nonnegative-drift case, we prove existence, uniqueness, and regularity of bounded classical solutions, and establish a verification theorem linking the value function and the HJB equation. Our second result reveals a sharp dichotomy in the structure of optimal controls: the push-the-laggard strategy is optimal for the all-survive criterion, but fails in general for the survivor-count criterion.

Related literature: The push-the-laggard strategy is an instance of the rank-dependent diffusions, which have been extensively studied in the literature, see e.g., [6, 7, 9, 12, 18, 20, 22, 25]. It has been proved to be the optimal drift control for the all-survive criterion [21] when $n = 2$, and for the survivor-count criterion [2, 26] when $n \to \infty$. See also [3] for recent work on drift control in high dimensions (i.e., n is large) for operations management problems. Besides, rank-dependent diffusions appeared in the study of stochastic games [17] and control of McKean-Vlasov dynamics [5, 10] in the context of financial systemic risk. Recently, [4, 8] studied the connection between rank-dependent diffusions and load balancing in queueing theory.

Organization of the paper: In the remainder of the introduction, we present the setup and the main results. In Section 2, we establish the connection between the control problems and the HJB equations (Proposition 1). We prove that the push-the-laggard strategy fails for the survivor-count criterion (Theorem 2(i)) in Section 3, while it holds for the all-survive criterion (Theorem 2(ii)) in Section 4. Technical lemmas are deferred to the appendix.

1.1 Problem formulation

We consider two allocation problems formulated on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and supporting infinitely many independent $\mathbb{F}$-Brownian motions $W^1, W^2, \ldots$. For every $n \geq 1$, denote by $\mathcal{A}_n$ the collection of $\mathbb{F}$-progressively measurable processes $\Phi = (\phi_t^1, \ldots, \phi_t^n)_{t\geq 0}$ taking values in $[0,1]^n$ and satisfying

$$\sum_{j=1}^{n} \phi_t^j \leq 1, \qquad \forall\, t \geq 0.$$

For each $\Phi \in \mathcal{A}_n$, define the controlled process $X^\Phi = (X_t^{\Phi,1}, \ldots, X_t^{\Phi,n})_{t\geq 0}$ by

$$\mathrm{d}X_t^{\Phi,i} := b\mathrm{d}t + \phi_t^i \mathrm{d}t + \mathrm{d}W_t^i, \qquad \forall\, t \geq 0,$$

and the hitting time

$$\tau_i^\Phi := \inf\left\{t \geq 0 :\ X_t^{\Phi,i} \leq 0\right\}.$$

We interpret $X_t^{\Phi,i}$ as the health, wealth, or reserve level of unit i: absorption at 0 represents failure, such as death, default, or bankruptcy. The control ϕ_t^i represents instantaneous assistance allocated to unit i, subject to a unit total budget[7].

We consider the following stochastic control problems. For $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, define

$$V^n(x) := \sup_{\Phi\in\mathcal{A}_n} \mathbb{E}\left[\prod_{i=1}^{n} \mathbb{1}_{\{\tau_i^\Phi = \infty\}} \,\middle|\, X_0^\Phi = x\right], \tag{1}$$

$$U^n(x) := \sup_{\Phi\in\mathcal{A}_n} \mathbb{E}\left[\sum_{i=1}^{n} \mathbb{1}_{\{\tau_i^\Phi = \infty\}} \,\middle|\, X_0^\Phi = x\right]. \tag{2}$$

The functional V^n corresponds to a system-reliability objective: maximize the probability that every unit survives forever. The functional U^n corresponds to a survivor-count objective: maximize the expected number of units that survive forever. These two objectives mirror, respectively, a no-one-left-behind policy and a save-as-many-as-possible policy in triage ethics [24, 27].

Another issue concerns the structure of optimal allocation. A natural conjecture is that, whenever one coordinate is strictly smaller than another, the value function is steeper in that coordinate, meaning that the

[7] All the results presented below remain valid if the unit budget is replaced by any positive budget.

marginal value of assistance is larger for the weakest unit. Equivalently, an optimal feedback would allocate effort to the smallest coordinate. This conjecture formalizes the question whether a prioritize-the-worst-off policy is optimal.

Define the push-the-laggard strategy $\overline{\Phi}$ by $\overline{\phi}_t^i := \mathbb{1}_{\{i=I_t\}}$, where

$$I_t := \min\{1 \le j \le n : \quad X_t^j = Z_t\} \qquad \text{and} \qquad Z_t := \min\{X_t^j : \quad X_t^j > 0,\ j = 1, \dots, n\}.$$

This system of SDEs has rank-dependent drifts and corresponds to an Atlas-type model whose existence and uniqueness are guaranteed by standard results on rank-based diffusions. Hence the conjecture reads:

$$\textbf{Conjecture* :} \;\; \text{Is } \overline{\Phi} \text{ optimal for } V^n(x) \text{ and for } U^n(x)?$$

1.2 Main result

Our answer is summarized in the theorem below.

Theorem 1. *Let $b \ge 0$ and $n \ge 2$. Then **Conjecture*** holds for V^n and does not hold for U^n.*

For $n = 1$, the unique optimizer is trivially given by $\phi^1 \equiv 1$, and

$$V^1(z) = U^1(z) = \mathbb{P}[z + (1+b)t + W_t^1 > 0, \forall t \ge 0] = H((b+1)^+ z), \quad \forall\, z \ge 0, \tag{3}$$

where $H : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$H(z) := 1 - \mathrm{e}^{-2z}.$$

Our analysis is built on the PDE counterpart of (1) and (2). It is worth noting that the parameter b plays a substantial role in the analysis of the corresponding HJB equation and in the study of the conjecture. Before turning to the PDE formulation, we record some basic properties of (1) and (2). Their proofs are postponed to the appendix.

Lemma 1. *As functions on $\mathbb{R}_+^n$, V^n and U^n satisfy the following properties.*

1. *V^n and U^n are symmetric on $\mathbb{R}_+^n$. For every $x \in \mathbb{R}_+^n$,*

$$0 \le V^n(x) \le \prod_{i=1}^n H((b+1)^+ x_i), \qquad 0 \le U^n(x) \le \sum_{i=1}^n H((b+1)^+ x_i).$$

   In particular, $V^n = 0$ on ∂E_n.

2. *The maps $\mathbb{R}_+ \ni x_i \longmapsto V^n(x_1, \dots, x_n) \in \mathbb{R}_+$ and $\mathbb{R}_+ \ni x_i \longmapsto U^n(x_1, \dots, x_n) \in \mathbb{R}_+$ are nondecreasing.*

Lemma 2. *For every $x \in E_n$, one has*

$$V^n(x) > 0 \quad \textit{if and only if} \quad b > -\frac{1}{n},$$
$$U^n(x) > 0 \quad \textit{if and only if} \quad b > -1.$$

In view of Lemma 2, the allocation problem is trivial when b is too negative, and it suffices to consider $b > -1/n$ for V^n and $b > -1$ for U^n. The next lemma provides the recursive relation between V^n and V^{n-1}, and between U^n and U^{n-1}. For later use, we adopt the notation that for $x \in \mathbb{R}^n$ and $i = 1, \dots, n$,

$$x^{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}, \qquad (x^{-i}, z) := (x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n) \in \mathbb{R}^n,$$
$$\max(x) := \max\{x_1, \dots, x_n\}, \qquad \min(x) := \min\{x_1, \dots, x_n\}.$$

Lemma 3. *Fix an arbitrary $i \in \{1, \dots, n\}$.*

1. *If $b \ge 0$ or $b \le -1/(n-1)$, then*
$$V^n(x^{-i}, \infty) = V^{n-1}(x^{-i}).$$
2. *If $b \ge 0$, then*
$$U^n(x^{-i}, \infty) = U^{n-1}(x^{-i}) + 1.$$

   Here we write $f(\infty) := \lim_{z \to \infty} f(z)$ whenever the limit exists. Both $V^n(x^{-i}, \infty)$ and $U^n(x^{-i}, \infty)$ are well defined by Lemma 1.

This paper studies **Conjecture*** by combining techniques from stochastic control and PDE analysis. Denote by $E_n := (0,\infty)^n$ the interior of $\mathbb{R}^n_+$ and by ∂E_n its boundary, namely

$$x \in \partial E_n := \big\{x \in \mathbb{R}^n_+ : \quad \min(x) = 0\big\}.$$

For $n \geq 2$, we write the corresponding HJB equation, with $\mathbf{1}_n := (1,\dots,1) \in \mathbb{R}^n$,

$$\frac{1}{2}\Delta p + b\mathbf{1}_n \cdot \nabla p + \max(\nabla p) = 0 \qquad \text{in } E_n, \tag{4}$$

together with suitable boundary conditions on ∂E_n:

$$p(x^{-i},0) = 0 \qquad \text{in } \partial E_n, \tag{5}$$

$$p(x^{-i},0) = U^{n-1}(x^{-i}) \qquad \text{in } \partial E_n. \tag{6}$$

The following proposition relates the stochastic control problems to the HJB equation.

Proposition 1. *Our results are stated separately for V^n and U^n, with $n \geq 2$.*

(i) *If $b \leq -1/n$, then* (4) *and* (5) *are well posed, and $V^n \equiv 0$ is their unique bounded classical solution.*

(ii) *If $b \leq -1$, then* (4) *and* (6) *are well posed, and $U^n \equiv 0$ is their unique bounded classical solution.*

(iii) *If $b \geq 0$, then* (4), (5) *and* (7) *are well posed, and $V^n \in C^2(E_n) \cap C(\mathbb{R}^n_+)$ is their unique bounded classical solution, where*

$$p(x^{-i},\infty) = V^{n-1}(x^{-i}). \tag{7}$$

(iv) *If $b \geq 0$, then* (4), (6) *and* (8) *are well posed, and $U^n \in C^2(E_n) \cap C(\mathbb{R}^n_+)$ is their unique bounded classical solution, where*

$$p(x^{-i},\infty) = U^{n-1}(x^{-i}) + 1. \tag{8}$$

Lemmas 2 and 3, together with Proposition 1, explain from both stochastic and PDE viewpoints the natural splitting of the drift parameter into three regimes:

– a trivial regime, where the value function vanishes identically;
– a nonnegative-drift regime, in which the recursion at infinity closes on the same $(n-1)$-dimensional problem;
– an intermediate negative-drift regime,

$$-\frac{1}{n} < b < 0 \text{ for } V^n, \qquad\qquad -1 < b < 0 \text{ for } U^n,$$

in which the value functions are nonzero, but the recursive boundary conditions at infinity no longer close on the same $(n-1)$-dimensional problem.

Thanks to the PDE characterization, we prove **Conjecture*** through the PDE formulation as follows: for every $x = (x_1,\dots,x_n) \in E_n$,

$$\textbf{Conjecture :}\ x_i = \min(x) \quad \Longrightarrow \quad \begin{cases} \partial_{x_i} V^n(x) = \max\big(\nabla V^n(x)\big), \\ \partial_{x_i} U^n(x) = \max\big(\nabla U^n(x)\big). \end{cases}$$

It is known that:

– For $b = 0$ and $n = 2$, McKean and Shepp gave in [21] the explicit expression

$$V^2(x) = 1 - \mathrm{e}^{-2\min(x)} - 2\min(x)\mathrm{e}^{-\mathbf{1}_2 \cdot x},$$

which proves the conjecture by direct verification. Moreover, they presented numerical evidence that the conjecture fails for U^2.

– For $b = 0$ and $n \longrightarrow \infty$, this conjecture is "asymptotically true" for U^n, as established by Tang and Tsai [26].

We have the following result.

Theorem 2. *Let $b \geq 0$ and $n \geq 2$.*

(i) ***Conjecture** holds for V^n.*

(ii) ***Conjecture** does not hold for U^n.*

Hence, in view of the ergodicity of n-dimensional Brownian motion and Itô's formula, Theorem 1 follows by combining Theorem 2 and Proposition 1.

For simplicity, we write $X^{\Phi,i} \equiv X^i$, $\tau_i^{\Phi} \equiv \tau_i$, and so on, and also $V^n \equiv V$, $U^n \equiv U$, and so on, whenever the context is clear.

1.3 On the intermediate negative-drift regime

In this subsection, we briefly discuss the intermediate negative-drift regimes

$$-\frac{1}{n} < b < 0 \quad \text{for } V^n, \qquad\qquad -1 < b < 0 \quad \text{for } U^n.$$

By Lemma 2, the value functions are strictly positive in these regimes. However, the recursive boundary conditions at infinity from Lemma 3 are no longer expected to hold in their present form, and the corresponding PDE analysis becomes more delicate.

First, the universal upper bounds

$$V^n(x^{-i}, z) \le V^{n-1}(x^{-i}), \qquad U^n(x^{-i}, z) \le 1 + U^{n-1}(x^{-i}), \qquad \forall\, z \ge 0, \tag{9}$$

follow directly from the definitions and hold for all $b \in \mathbb{R}$. The difficulty is to identify the correct matching lower bounds, and hence the correct asymptotic trace at infinity.

Indeed, if the i-th coordinate is started from a large level z, then in order for it to survive with probability tending to one, one must assign to it a positive effective drift. If one freezes a constant control $\eta \in [0, 1]$ on the i-th coordinate, then

$$\mathbb{P}\big[\tau_i = \infty \,\big|\, X_0^i = z\big] = H((b+\eta)^+ z).$$

Hence, when $b < 0$, asymptotic survival of the remote coordinate requires $\eta > -b$, so a nonvanishing fraction of the total budget must be reserved for it. Consequently, the remaining $n-1$ coordinates do not asymptotically inherit the full unit budget.

To formalize this observation, for $a \in [0, 1]$, let $V^{n-1,a}$ and $U^{n-1,a}$ denote the $(n-1)$-dimensional value functions associated with the same dynamics, but with total budget a in place of 1, namely

$$V^{n-1,a}(y) := \sup_{\Psi} \mathbb{E}\Big[\prod_{j=1}^{n-1} \mathbb{1}_{\{\tau_j^{\Psi} = \infty\}} \,\Big|\, X_0^{\Psi} = y\Big],$$

and

$$U^{n-1,a}(y) := \sup_{\Psi} \mathbb{E}\Big[\sum_{j=1}^{n-1} \mathbb{1}_{\{\tau_j^{\Psi} = \infty\}} \,\Big|\, X_0^{\Psi} = y\Big],$$

where the supremum is taken over all progressively measurable controls $\Psi = (\psi^1, \ldots, \psi^{n-1})$ satisfying $\psi_t^j \ge 0$ and

$$\sum_{j=1}^{n-1} \psi_t^j \le a, \qquad t \ge 0.$$

Then, for every $\eta \in (-b, 1]$, freezing $\phi^i \equiv \eta$ on the remote coordinate yields

$$\liminf_{z\to\infty} V^n(x^{-i}, z) \ge V^{n-1,\,1-\eta}(x^{-i}), \tag{10}$$

and

$$\liminf_{z\to\infty} U^n(x^{-i}, z) \ge 1 + U^{n-1,\,1-\eta}(x^{-i}). \tag{11}$$

Since the maps $a \mapsto V^{n-1,a}$ and $a \mapsto U^{n-1,a}$ are nondecreasing, these estimates suggest that the correct asymptotic traces at infinity should be governed by the reduced budget $1+b$. More precisely, one is naturally led to conjecture that

$$\lim_{z\to\infty} V^n(x^{-i}, z) = V^{n-1,\,1+b}(x^{-i}), \tag{12}$$

and

$$\lim_{z\to\infty} U^n(x^{-i}, z) = 1 + U^{n-1,\,1+b}(x^{-i}). \tag{13}$$

At present, however, we do not have a matching upper-bound argument proving these identities. Establishing such an upper bound appears to require a genuinely new argument showing that any asymptotically optimal strategy must devote at least $-b$ units of budget to the remote coordinate.

This also clarifies the PDE situation. Consider first the case of V^n with $-1/n < b < 0$. The pair consisting of the interior HJB equation (4) and the finite-boundary condition (5) is not well posed in the class of bounded solutions: the zero function is a bounded classical solution, while Lemma 2 implies that the stochastic control value function V^n is strictly positive on E_n. Hence uniqueness fails if one prescribes only the boundary condition on ∂E_n and does not specify the behaviour at infinity. The same remark applies to U^n in the regime $-1 < b < 0$, where again the corresponding HJB problem with only the boundary condition (6) is not expected to be well posed.

Although the asymptotic boundary condition at infinity is unclear in the intermediate negative-drift regime, the stochastic control problems themselves are unchanged, and the same weak dynamic-programming argument still leads formally to the HJB equation (4) in E_n. For instance, in the standard viscosity sense, the upper semicontinuous envelope of V^n is a viscosity subsolution of (4) on E_n, while the lower semicontinuous envelope of V^n is a viscosity supersolution of (4) on E_n. The main obstacle is different: without an identified boundary condition at infinity, the present method does not provide continuity of V^n or U^n on E_n, although their continuity on ∂E_n is clear from the control problem. Without continuity in E_n, the later regularity argument cannot be applied in the same way.

Finally, let us emphasize that the missing ingredient is global, not local. If continuity of V^n or U^n in E_n could be established by another argument, then the local elliptic regularity part of the proof, as shown in Section 2, should still apply to obtain $V^n, U^n \in C^2(E_n)$. In other words, the principal obstruction in the intermediate negative-drift regime is the lack of an identified trace at infinity and the resulting failure of global comparison and uniqueness, rather than the local regularity theory itself.

2 Proof of Proposition 1

This section is devoted to the proof of Proposition 1. We begin by recalling some basic properties of the value functions, and then distinguish two cases according to the sign of the drift parameter b. For $n \geq 2$, we write the HJB equation in the form

$$F(\nabla p, \nabla^2 p) = 0 \qquad \text{in } E_n, \tag{14}$$

where the Hamiltonian $F : \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ is defined by

$$F(q, \Gamma) := \frac{1}{2}\operatorname{tr}(\Gamma) + b\,\mathbf{1}_n \cdot q + \max(q), \qquad \text{with } \mathbb{S}^n := \{\Gamma = (\gamma_{ij})_{1\leq i,j\leq n} : \gamma_{ij} = \gamma_{ji} \in \mathbb{R}\}.$$

2.1 The case of sufficiently negative drift

By Lemma 2, one has $V^n \equiv 0$ when $b \leq -1/n$ and $U^n \equiv 0$ when $b \leq -1$. So it remains to show the uniqueness, summarized in the proposition below.

Proposition 2. *Let $v^n \in C^2(E_n) \cap C(\mathbb{R}^n_+)$ be a bounded solution of* (4) *and* (5)*. If $b \leq -1/n$, then $v^n \equiv 0$. Let $u^n \in C^2(E_n) \cap C(\mathbb{R}^n_+)$ be a bounded solution of* (4) *and* (6)*. If $b \leq -1$, then $u^n \equiv 0$.*

Proposition 2 yields parts (i) and (ii) of Proposition 1. Its proof, similar to that of Theorem 2, also follows from the verification argument and is postponed in Appendix.

2.2 The case $b \geq 0$

We assume $b \geq 0$ throughout this subsection. We first prove the result for V^n, and the proof for U^n is identical up to the corresponding boundary data. We proceed by induction on n. For $n = 1$, the claim is immediate, since (4) reduces to an ODE and

$$V^1(z) = H((b+1)z) \in C^2(E_1) \cap C(\mathbb{R}_+)$$

is the unique bounded classical solution. Assume now that the statement has been proved up to dimension $n-1$. For the sake of presentation, we write $V \equiv V^n$ without any danger of confusion.

Recall that

$$V(x) = \sup_{\Phi \in \mathcal{A}_n} \mathbb{E}\Big[\prod_{i=1}^{n} \mathbb{1}_{\{\tau_i^\Phi = \infty\}} \Big| X_0^\Phi = x\Big], \qquad x \in \mathbb{R}^n_+,$$

and denote by V^* and V_* its upper and lower semicontinuous envelopes on E_n. By the weak dynamic-programming principle of Bouchard and Touzi [11], which applies to the present controlled diffusion setting, V^* is a viscosity subsolution and V_* is a viscosity supersolution of (14) on E_n.

The following proposition shows the continuity of V, which is the key to establish higher regularity of V.

Proposition 3. *One has $V^* \leq V_*$ on E_n. Consequently, $V = V^* = V_*$ is continuous on $\mathbb{R}^n_+$ and is a viscosity solution of* (14) *on E_n.*

The idea is to transform the comparison argument to a bounded domain. Namely, we introduce a coordinatewise compactification. Let $f:(0,1)\to(0,\infty)$ be defined by

$$f(z):=\frac{z}{1-z},\qquad z\in(0,1),$$

and extend it componentwise to a homeomorphism, still denoted by $f:J_n:=(0,1)^n\longrightarrow E_n$,

$$f(y):=\Big(\frac{y_1}{1-y_1},\dots,\frac{y_n}{1-y_n}\Big).$$

Its inverse is the componentwise map

$$g(x):=\frac{x}{1+x},\qquad x\in(0,\infty).$$

Define $\widetilde V:=V\circ f$, $\widetilde V^*:=V^*\circ f$ and $\widetilde V_*:=V_*\circ f$, and set for $y\in J_n$:

$$a_i(y):=(1-y_i)^4,\qquad \beta_i(y):=(1-y_i)^2\big(b-(1-y_i)\big),\qquad \gamma_i(y):=(1-y_i)^2.$$

Define the alternative Hamiltonian $\mathcal H:J_n\times\mathbb R^n\times\mathbb S^n\to\mathbb R$ by

$$\mathcal H(y,q,\Gamma):=\frac12\sum_{i=1}^n a_i(y)\,\gamma_{ii}+\sum_{i=1}^n\beta_i(y)\,q_i+\max_{1\le i\le n}\gamma_i(y)\,q_i. \tag{15}$$

Lemma 4. *The function $\widetilde V^*$ is a bounded upper semicontinuous viscosity subsolution of*

$$\mathcal H(y,\nabla u,\nabla^2u)=0\qquad \text{in } J_n,$$

and $\widetilde V_*$ is a bounded lower semicontinuous viscosity supersolution of the same equation.

Proof. This is the standard invariance of viscosity inequalities under C^2 diffeomorphisms, applied to the map $f:J_n\to E_n$. The specific form of (15) is obtained by the chain rule. □

We next identify the boundary values induced by the absorbing boundary and the recursion at infinity. For $y\in\partial J_n$, set

$$I_0(y):=\{i:\ y_i=0\},\qquad I_1(y):=\{i:\ y_i=1\},\qquad m(y):=|I_1(y)|.$$

If $m(y)<n$, let $y|_{(I_1(y))^c}$ denote the $(n-m(y))$-tuple obtained by deleting the coordinates equal to 1. With the convention $V^0\equiv1$, define $G:\partial J_n\to[0,1]$ by

$$G(y):=\begin{cases}0, & \text{if } I_0(y)\neq\emptyset,\\ V^{\,n-m(y)}\big(f(y|_{(I_1(y))^c})\big), & \text{if } I_0(y)=\emptyset.\end{cases} \tag{16}$$

Lemma 5. *One has $\widetilde V^*=G=\widetilde V_*$ on ∂J_n.*

Proof. If $I_0(y)\neq\emptyset$, then $f(y)\in\partial E_n$, and hence $V(f(y))=0$. Therefore

$$\widetilde V^*(y)=\widetilde V_*(y)=0=G(y).$$

Assume now that $I_0(y)=\emptyset$. Then $m(y)\ge1$, and along every sequence $y^k\to y$ with $y^k\in J_n$, one has $f(y^k)_i\to\infty$ for $i\in I_1(y)$ while the remaining coordinates converge to $f(y|_{(I_1(y))^c})$. By Lemma 3, the corresponding limit of V is precisely

$$V^{n-m(y)}\big(f(y|_{(I_1(y))^c})\big)=G(y).$$

This gives $\widetilde V^*(y)\le G(y)\le\widetilde V_*(y)$. Since always $\widetilde V_*\le\widetilde V^*$, the conclusion follows. □

Now we are ready to prove Proposition 3.

Proof (of Proposition 3). By Lemma 4, $\widetilde{V}^*$ is a bounded upper semicontinuous viscosity subsolution and $\widetilde{V}_*$ is a bounded lower semicontinuous viscosity supersolution of the compactified equation on J_n. By Lemma 5,

$$\widetilde{V}^* \le \widetilde{V}_* \qquad \text{on } \partial J_n.$$

Since the operator $\mathcal{H}$ is continuous, degenerate elliptic, and J_n is bounded, the standard viscosity comparison theorem applies, yielding

$$\widetilde{V}^* \le \widetilde{V}_* \qquad \text{on } J_n.$$

Because always $\widetilde{V}_* \le \widetilde{V} \le \widetilde{V}^*$, it follows that

$$\widetilde{V} = \widetilde{V}^* = \widetilde{V}_* \qquad \text{on } J_n.$$

Composing with $g = f^{-1}$ gives $V = V^* = V_*$ on E_n. Thus V is continuous on E_n, and thus on $\mathbb{R}^n_+$ by Lemma 1. In particular, V is a viscosity solution of (14). □

We now derive local regularity. The argument itself is standard in interior elliptic regularity theory. However, since our equation is posed on the unbounded domain E_n and we could not find a convenient reference stated exactly in this form. For the convenience of the reader, we include the proof.

Lemma 6. *Let $K \Subset E_n$ be compact. Then there exists $C_K > 0$ such that*

$$|V(x) - V(y)| \le C_K |x - y|, \qquad \forall x, y \in K.$$

Proof. The Hamiltonian F is degenerate elliptic and has no explicit dependence on the space variable or on the unknown itself. Moreover, it satisfies the structural assumptions of Ishii–Lions [19, Sec. VII.1]. Since $0 \le V \le 1$ on E_n, the interior Lipschitz regularity theorem of Ishii–Lions [19, Thm. VII.2] applies and yields the desired estimate. □

We now turn to higher interior regularity. We briefly indicate the strategy and postpone the details to the appendix.

With the interior Lipschitz continuity of V ensured by Lemma 6, V is differentiable almost everywhere by Rademacher's theorem. So the nonlinear term

$$G(\nabla V) := -b\,\mathbf{1}_n \cdot \nabla V - \max(\nabla V)$$

is well defined almost everywhere in $\mathbb{R}^n_+$. Since G is globally Lipschitz, one may rewrite the HJB equation in the form

$$\Delta V = 2G(\nabla V).$$

The proof then proceeds in three steps. First, one shows that the identity

$$\Delta V = 2G(\nabla V)$$

holds in the distributional sense on compact subsets of E_n. Second, interior Calderón–Zygmund estimates imply that

$$V \in W^{2,p}_{\text{loc}}(E_n) \qquad \text{for every } p \in (1, \infty).$$

Third, Sobolev embedding and interior Schauder estimates yield

$$V \in C^{2,\alpha}_{\text{loc}}(E_n) \qquad \text{for some } \alpha \in (0,1).$$

For the convenience of the reader, we recall the relevant local function spaces and give a detailed proof in the appendix.

Lemma 7. *For every $p \in (1, \infty)$, one has $V \in W^{2,p}_{\text{loc}}(E_n)$. Consequently, there exists $\alpha \in (0,1)$ such that $V \in C^{2,\alpha}_{\text{loc}}(E_n)$, and in particular $V \in C^2(E_n) \cap C(\mathbb{R}^n_+)$.*

Finally, we conclude part (iii) of Proposition 1 by the lemma below.

Lemma 8. *Let $v \in C^2(E_n) \cap C(\mathbb{R}^n_+)$ be a bounded classical solution of* (4)*,* (5)*, and* (7)*. Then $v = V$.*

Its proof is the same as that of Proposition 2, and is therefore omitted.

Remark 1. The proof for U^n is identical. One replaces the boundary function G in (16) by

$$\widehat{G}(y) := \begin{cases} U^{n-1}\big(f(y|_{\{i:\, y_i = 0\}^c})\big), & \text{if } I_0(y) \neq \emptyset, \\ m(y) + U^{n-m(y)}\big(f(y|_{(I_1(y))^c})\big), & \text{if } I_0(y) = \emptyset, \end{cases}$$

and the same comparison and regularity arguments apply. This proves part (iv) of Proposition 1.

3 Proof of Theorem 2: Conjecture fails for U^n

This section is devoted to the proof of Theorem 2 (ii), namely the failure of **Conjecture** for U^n. We proceed in two steps.

- First, we treat the two-dimensional case by deriving a contradiction from a mixed Dirichlet–Neumann corner estimate near the origin.
- Then we lift the two-dimensional failure to all dimensions by combining the boundary condition at infinity (8) with a compactness argument.

3.1 The two-dimensional counterexample

Theorem 3. *Write $u \equiv U^2$. Then there exists a nonempty open set*

$$O \subset \{(x_1, x_2) \in E_2 :\ x_1 < x_2\}$$

such that

$$\partial_{x_2} u(x_1, x_2) > \partial_{x_1} u(x_1, x_2), \qquad \forall (x_1, x_2) \in O.$$

Proof. Let

$$D := \{(x_1, x_2) \in E_2 :\ x_1 < x_2\}.$$

We argue by contradiction and assume that

$$\partial_{x_1} u(x_1, x_2) \geq \partial_{x_2} u(x_1, x_2), \qquad \forall (x_1, x_2) \in D. \tag{17}$$

Step 1. Under (17), one has

$$\max\{\partial_{x_1} u, \partial_{x_2} u\} = \partial_{x_1} u \qquad \text{on } D.$$

Hence u solves the linear equation

$$\frac{1}{2} \Delta u + (b+1) \partial_{x_1} u + b \partial_{x_2} u = 0 \qquad \text{in } D. \tag{18}$$

Step 2. The boundary of D inside $\mathbb{R}^2_+$ consists of the two rays

$$\Gamma_D := \{(0, x_2) : x_2 > 0\}, \qquad \Gamma_N := \{(z, z) : z > 0\}.$$

On the Dirichlet part Γ_D, the boundary condition (6) gives

$$u(0, x_2) = U^1(x_2) = 1 - \mathrm{e}^{-2(b+1)x_2}, \qquad x_2 > 0. \tag{19}$$

On the diagonal Γ_N, the symmetry of u implies

$$\partial_{x_1} u(z, z) = \partial_{x_2} u(z, z), \qquad z > 0.$$

If n denotes the inward unit normal to D along Γ_N, proportional to $(-1, 1)$, then

$$\partial_n u = 0 \qquad \text{on } \Gamma_N. \tag{20}$$

Therefore u solves the mixed boundary value problem

$$\begin{cases} \frac{1}{2} \Delta u + (b+1) \partial_{x_1} u + b \partial_{x_2} u = 0 & \text{in } D, \\ u = U^1 & \text{on } \Gamma_D, \\ \partial_n u = 0 & \text{on } \Gamma_N. \end{cases} \tag{21}$$

Step 3. Since

$$U^1(z) = 1 - \mathrm{e}^{-2(b+1)z} = 2(b+1)z + O(z^2) \qquad \text{as } z \downarrow 0,$$

we introduce the affine function

$$\bar{u}(x_1, x_2) := -2b\, x_1 + 2(b+1)\, x_2. \tag{22}$$

Then $\Delta \bar{u} = 0$, and

$$\frac{1}{2}\Delta\bar{u} + (b+1)\partial_{x_1}\bar{u} + b\partial_{x_2}\bar{u} = (b+1)(-2b) + b\cdot 2(b+1) = 0.$$

So $\bar{u}$ solves the same interior equation as u. Moreover,

$$\bar{u}(0,x_2) = 2(b+1)x_2,$$

which matches the boundary datum $U^1(x_2)$ to first order at the origin.

Set

$$H := u - \bar{u}.$$

Then H solves

$$\frac{1}{2}\Delta H + (b+1)\partial_{x_1}H + b\partial_{x_2}H = 0 \qquad \text{in } D, \tag{23}$$

with boundary conditions

$$H(0,x_2) = u(0,x_2) - 2(b+1)x_2 = O(x_2^2) \qquad \text{as } x_2 \downarrow 0, \tag{24}$$

$$\partial_n H = -\partial_n \bar{u} \qquad \text{on } \Gamma_N. \tag{25}$$

Since $\bar{u}$ is affine, the Neumann datum in (25) is constant.

Step 4. We now apply the standard Zaremba corner estimate.

Write points in polar coordinates:

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta.$$

Then

$$D = \{x_1 > 0,\ x_2 > x_1\}$$

corresponds to

$$\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right),$$

so the opening angle of the wedge is

$$\omega = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

The principal part of the operator in (23) is $\frac{1}{2}\Delta$. For the Laplacian in a wedge with mixed Dirichlet–Neumann boundary conditions, separated solutions of the form

$$r^\lambda \Phi(\theta)$$

lead to the angular problem

$$\Phi''(\theta) + \lambda^2\Phi(\theta) = 0, \qquad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right),$$

with boundary conditions

$$\Phi'\left(\frac{\pi}{4}\right) = 0, \qquad \Phi\left(\frac{\pi}{2}\right) = 0.$$

Its singular exponents are

$$\lambda_k = \frac{(2k+1)\pi}{2\omega} = 2 + 4k, \qquad k = 0,1,2,\ldots,$$

so the smallest positive singular exponent is $\lambda_0 = 2$.

Since the Dirichlet datum in (24) vanishes to order 2 at the corner and the Neumann datum in (25) is bounded, the Kondrat'ev–Grisvard–Dauge theory for mixed boundary value problems in planar corners implies that there exist $\rho > 0$ and $C > 0$ such that

$$|\nabla H(x)| \le C|x|, \qquad \forall x \in D \text{ with } |x| < \rho. \tag{26}$$

See, for example, Grisvard [**?**, Ch. 4, §4.4] and Dauge [14, Ch. 6].

Step 5. Since $u = \bar{u} + H$, by (22) we have

$$\partial_{x_1}u = -2b + \partial_{x_1}H, \qquad \partial_{x_2}u = 2(b+1) + \partial_{x_2}H.$$

Hence, by (26), for $|x| < \rho$,

$$|\partial_{x_1}u(x) + 2b| \le C|x|, \qquad |\partial_{x_2}u(x) - 2(b+1)| \le C|x|.$$

Choose $0 < r_0 < \rho$ such that $Cr_0 < 1$, and define

$$O := \{x \in D :\ |x| < r_0\}.$$

Then O is a nonempty open subset of D, and for every $x \in O$,

$$\partial_{x_2}u(x) \ge 2(b+1) - Cr_0 > 2b+1 \ge -2b + Cr_0 \ge \partial_{x_1}u(x).$$

This contradicts (17). Therefore (17) cannot hold on all of D, and the theorem follows. □

3.2 A dimension-lifting argument

The next lemma provides the key step for lifting the two-dimensional counterexample to all higher dimensions.

Lemma 9. *Fix $m \geq 2$. For $R > 0$, define the function*

$$w_R(x) := U^{m+1}(x, R) - 1, \qquad x \in \mathbb{R}_+^m.$$

Then

$$w_R \longrightarrow U^m \qquad \text{in } C^2_{\mathrm{loc}}(E_m) \qquad \text{as } R \to \infty.$$

Proof. Fix a compact set $K \Subset E_m$, and choose another compact set $K' \Subset E_m$ such that

$$K \Subset \operatorname{int}(K').$$

Set

$$\delta := \operatorname{dist}(K, \partial K') > 0.$$

Since $K' \Subset E_m$, there exists $\delta_0 > 0$ such that every point of K' lies at distance at least δ_0 from ∂E_m.

For $R \geq 1$, define the lifted compact sets

$$\widetilde{K}_R := \{(x, R) : x \in K\}, \qquad \widetilde{K}'_R := \{(x, R) : x \in K'\}.$$

Then every point of $\widetilde{K}'_R$ lies at distance at least

$$\min(\delta_0, R) \geq \min(\delta_0, 1) > 0$$

from ∂E_{m+1}. In particular, the distance from $\widetilde{K}_R$ to the boundary of E_{m+1} is bounded from below uniformly in $R \geq 1$.

Now U^{m+1} solves

$$\frac{1}{2}\Delta u + b\,\mathbf{1}_{m+1} \cdot \nabla u + \max(\nabla u) = 0 \qquad \text{in } E_{m+1},$$

and satisfies

$$0 \leq U^{m+1} \leq m + 1 \qquad \text{on } E_{m+1}.$$

Since the equation has constant principal part, is uniformly elliptic, and has bounded lower-order coefficients, the standard interior Schauder estimate applies on neighborhoods of $\widetilde{K}_R$, with constants depending only on the equation, the Hölder exponent, and the above uniform lower bound on the distance to the boundary. Therefore there exist $\alpha \in (0, 1)$ and a constant $C_K > 0$, independent of R, such that

$$\|U^{m+1}\|_{C^{2,\alpha}(\widetilde{K}_R)} \leq C_K, \qquad \forall R \geq 1.$$

Since

$$w_R(x) = U^{m+1}(x, R) - 1, \qquad x \in E_m,$$

this implies

$$\|w_R\|_{C^{2,\alpha}(K)} \leq C_K, \qquad \forall R \geq 1.$$

By the compact embedding

$$C^{2,\alpha}(K) \hookrightarrow C^2(K),$$

the family $(w_R)_{R \geq 1}$ is precompact in $C^2(K)$. Since $K \Subset E_m$ was arbitrary, a diagonal argument shows that from every sequence $R_\ell \to \infty$ one may extract a subsequence, still denoted by (R_ℓ), and a function $w \in C^2(E_m)$ such that

$$w_{R_\ell} \to w \qquad \text{in } C^2_{\mathrm{loc}}(E_m).$$

It remains to identify the limit. By the boundary condition at infinity (8) in dimension $m + 1$, for every fixed $x \in E_m$,

$$\lim_{R \to \infty} U^{m+1}(x, R) = 1 + U^m(x).$$

Equivalently,

$$\lim_{R \to \infty} w_R(x) = U^m(x), \qquad x \in E_m.$$

Hence every subsequential limit w must coincide pointwise with U^m. Therefore

$$w = U^m \qquad \text{on } E_m.$$

Since every subsequence has the same limit, the whole family converges:

$$w_R \to U^m \qquad \text{in } C^2_{\mathrm{loc}}(E_m) \qquad \text{as } R \to \infty.$$

This completes the proof. □

3.3 Failure for all $n \geq 2$

Theorem 4. *Fix $b \geq 0$ and $n \geq 2$. Then* ***Conjecture*** *fails for U^n: there exists $x \in E_n$ such that*

$$x_1 = \min(x) \qquad \text{and} \qquad \partial_{x_2} U^n(x) > \partial_{x_1} U^n(x).$$

Moreover, the strict inequality holds on a nonempty open subset of

$$\{x \in E_n : \ x_1 < x_2\}.$$

Proof. We argue by induction on n.

For $n = 2$, the result is exactly Theorem 3.

Assume now that the statement holds in dimension $m \geq 2$. Then there exists $x^* \in E_m$ such that

$$x_1^* = \min(x^*)$$

and

$$\partial_{x_2} U^m(x^*) - \partial_{x_1} U^m(x^*) \geq 2\varepsilon \tag{27}$$

for some $\varepsilon > 0$. We show that the statement then holds in dimension $m + 1$.

Choose a compact set $K \Subset E_m$ containing x^* in its interior. By Lemma 9,

$$U^{m+1}(\cdot, R) - 1 \longrightarrow U^m \qquad \text{in } C^2(K) \qquad \text{as } R \to \infty.$$

In particular, for $i = 1, 2$,

$$\partial_{x_i} U^{m+1}(x^*, R) \longrightarrow \partial_{x_i} U^m(x^*) \qquad \text{as } R \to \infty.$$

Hence, by (27), for R large enough one has

$$\partial_{x_2} U^{m+1}(x^*, R) - \partial_{x_1} U^{m+1}(x^*, R) \geq \varepsilon.$$

Choose such an R moreover satisfying $R > \max(x^*)$, and define

$$\widehat{x} := (x^*, R) \in E_{m+1}.$$

Then

$$\widehat{x}_1 = \min(\widehat{x})$$

and

$$\partial_{x_2} U^{m+1}(\widehat{x}) > \partial_{x_1} U^{m+1}(\widehat{x}).$$

This proves that **Conjecture** fails in dimension $m + 1$.

The conclusion for all $n \geq 2$ follows by induction. Finally, since $U^n \in C^2(E_n)$, the strict inequality at one point implies the same inequality on a sufficiently small neighborhood of that point. Therefore it holds on a nonempty open subset of $\{x \in E_n : \ x_1 < x_2\}$. □

4 Proof of Theorem 2: Conjecture holds for V^n

For convenience, we recall two basic properties of V^n:

- V^n is symmetric in its coordinates;
- V^n is componentwise nondecreasing on $\mathbb{R}_+^n$.

To prove **Conjecture**, it is enough to establish the following pairwise comparison.

Theorem 5. *Let $b \geq 0$. Then for all $i \neq j$ and all $x \in E_n$ such that $x_i < x_j$,*

$$\partial_{x_i} V^n(x) \geq \partial_{x_j} V^n(x).$$

Clearly, Theorem 5 implies Theorem 2. By symmetry, it is enough to prove the statement for the pair $(i, j) = (1, 2)$, namely

$$x_1 < x_2 \qquad \Longrightarrow \qquad \partial_{x_1} V^n(x) \geq \partial_{x_2} V^n(x).$$

4.1 Difference quotient on the wedge

Define

$$D := \{x \in E_n : \; x_1 < x_2\}.$$

Fix $h > 0$, and for $x \in D$ define

$$u(x) := V^n(x_1 + h, x_2, x_3, \dots, x_n), \qquad \widetilde{u}(x) := V^n(x_1, x_2 + h, x_3, \dots, x_n),$$

and

$$w_h(x) := u(x) - \widetilde{u}(x).$$

Since $V^n \in C^2(E_n)$ solves (4), both u and $\widetilde{u}$ solve the same HJB equation on D. Subtracting their equations yields

$$\frac{1}{2}\Delta w_h + b\sum_{i=1}^{n} \partial_{x_i} w_h + \big(\max(\nabla u) - \max(\nabla \widetilde{u})\big) = 0 \qquad \text{in } D. \tag{28}$$

Since the map $q \mapsto \max(q)$ is convex, for each $x \in D$ there exists $\lambda(x) = (\lambda_1(x), \dots, \lambda_n(x)) \in [0,1]^n$ with $\sum_k \lambda_k(x) = 1$ such that

$$\max(\nabla u(x)) - \max(\nabla \widetilde{u}(x)) = \sum_{k=1}^{n} \lambda_k(x)\, \partial_{x_k} w_h(x).$$

Hence w_h solves the linear equation

$$\frac{1}{2}\Delta w_h + \sum_{k=1}^{n} c_k(x)\, \partial_{x_k} w_h = 0 \qquad \text{in } D, \tag{29}$$

where

$$c_k(x) := b + \lambda_k(x) \in [b, b+1].$$

In particular, (29) is a linear uniformly elliptic equation with bounded measurable drift coefficients.

4.2 Boundary values of w_h

We next identify the sign of w_h on the finite boundary of D.

(i) The diagonal $x_1 = x_2$. If $x_1 = x_2 = z > 0$, then

$$w_h(z, z, x_3, \dots, x_n) = V^n(z + h, z, x_3, \dots, x_n) - V^n(z, z + h, x_3, \dots, x_n) = 0$$

by symmetry of V^n in the first two coordinates.

(ii) The face $x_1 = 0$. If $x_1 = 0$, then

$$w_h(0, x_2, x_3, \dots, x_n) = V^n(h, x_2, x_3, \dots, x_n) - V^n(0, x_2 + h, x_3, \dots, x_n).$$

The second term vanishes because $(0, x_2 + h, \dots) \in \partial E_n$, while the first term is nonnegative. Thus

$$w_h(0, x_2, x_3, \dots, x_n) \geq 0.$$

(iii) The faces $x_k = 0$ for $k \geq 3$. If $x_k = 0$ for some $k \in \{3, \dots, n\}$, then both arguments of V^n lie on ∂E_n, and therefore

$$w_h(x) = 0.$$

Hence

$$w_h \geq 0 \qquad \text{on the finite boundary of } D. \tag{30}$$

4.3 Barrier argument and conclusion

Define the linear operator

$$L\phi := \frac{1}{2}\Delta\phi + \sum_{k=1}^{n} c_k(x)\,\partial_{x_k}\phi.$$

Then (29) becomes

$$Lw_h = 0 \qquad \text{in } D.$$

Now introduce

$$\Phi(x) := x_1 + \cdots + x_n, \qquad u_\delta(x) := w_h(x) + \delta\Phi(x), \qquad \delta > 0.$$

Since $\Delta\Phi = 0$ and $\partial_{x_k}\Phi = 1$, we have

$$L\Phi = \sum_{k=1}^{n} c_k(x) = nb + 1 \geq 1.$$

Therefore

$$Lu_\delta = Lw_h + \delta L\Phi = \delta(nb+1) \geq \delta > 0 \qquad \text{in } D.$$

On the finite boundary of D, (30) implies

$$u_\delta = w_h + \delta\Phi \geq 0.$$

Moreover, since $0 \leq V^n \leq 1$, one has $|w_h| \leq 1$, so

$$u_\delta(x) \geq -1 + \delta\Phi(x).$$

As $|x| \to \infty$ in D, one has $\Phi(x) \to \infty$, hence

$$u_\delta(x) \to \infty.$$

It follows that u_δ attains its minimum at some point $x^* \in \overline{D}$.

We claim that this minimum is nonnegative. Indeed:

- if $x^* \in \partial D \cap \mathbb{R}^n_+$, then $u_\delta(x^*) \geq 0$ by the boundary condition above; - if $x^* \in D$, then $u_\delta \in C^2(D)$ satisfies $Lu_\delta \geq 0$ and attains an interior minimum. By the strong maximum principle for linear uniformly elliptic equations with bounded coefficients, u_δ must be constant on D. Since $u_\delta \geq 0$ on the finite boundary, this constant must be nonnegative.

Hence in all cases

$$u_\delta(x) \geq 0, \qquad \forall x \in D.$$

That is,

$$w_h(x) \geq -\delta\Phi(x), \qquad \forall x \in D.$$

Letting $\delta \downarrow 0$ gives

$$w_h(x) \geq 0, \qquad \forall x \in D.$$

By definition of w_h, this means

$$V^n(x_1 + h, x_2, x_3, \ldots, x_n) \geq V^n(x_1, x_2 + h, x_3, \ldots, x_n), \qquad \forall x \in D,\ \forall h > 0.$$

Subtracting $V^n(x)$ from both sides, dividing by h, and letting $h \downarrow 0$, we obtain

$$\partial_{x_1} V^n(x) \geq \partial_{x_2} V^n(x), \qquad \forall x \in D.$$

This proves Theorem 5, and therefore Theorem 2 (i).

A Appendix of related proofs

Proof (of Lemma 1). We prove the stated properties separately.

(i) For every admissible control $\Phi \in \mathcal{A}_n$ and every $i = 1, \ldots, n$, one has

$$X_t^{\Phi,i} = x_i + \int_0^t (b + \phi_s^i)\, \mathrm{d}s + W_t^i \le x_i + (b+1)t + W_t^i, \qquad t \ge 0,$$

since $0 \le \phi_t^i \le 1$. Hence,

$$\{\tau_i^\Phi = \infty\} \subset \Big\{x_i + (b+1)t + W_t^i > 0,\ \forall t \ge 0\Big\} \quad \Longrightarrow \quad \mathbb{P}[\tau_i^\Phi = \infty \mid X_0^\Phi = x] \le H((b+1)^+ x_i).$$

Since the Brownian motions are independent, it follows that

$$\mathbb{E}\Big[\prod_{i=1}^n \mathbb{1}_{\{\tau_i^\Phi = \infty\}} \,\Big|\, X_0^\Phi = x\Big] \le \prod_{i=1}^n H((b+1)^+ x_i), \qquad \mathbb{E}\Big[\sum_{i=1}^n \mathbb{1}_{\{\tau_i^\Phi = \infty\}} \,\Big|\, X_0^\Phi = x\Big] \le \sum_{i=1}^n H((b+1)^+ x_i).$$

Taking the supremum over $\Phi \in \mathcal{A}_n$ yields

$$0 \le V^n(x) \le \prod_{i=1}^n H((b+1)^+ x_i), \qquad 0 \le U^n(x) \le \sum_{i=1}^n H((b+1)^+ x_i).$$

(ii) Fix $i \in \{1, \ldots, n\}$ and let $x, y \in \mathbb{R}_+^n$ be such that $x_j = y_j$ for $j \ne i$ and $x_i \le y_i$. For any admissible control Φ, the corresponding controlled processes satisfy

$$X_t^{\Phi,j,y} = X_t^{\Phi,j,x}, \quad j \ne i, \qquad X_t^{\Phi,i,y} = X_t^{\Phi,i,x} + (y_i - x_i), \quad t \ge 0.$$

Thus $\tau_j^{\Phi,y} = \tau_j^{\Phi,x}$ for $j \ne i$, while $\tau_i^{\Phi,y} \ge \tau_i^{\Phi,x}$. Hence

$$\prod_{j=1}^n \mathbb{1}_{\{\tau_j^{\Phi,x} = \infty\}} \le \prod_{j=1}^n \mathbb{1}_{\{\tau_j^{\Phi,y} = \infty\}}, \qquad \sum_{j=1}^n \mathbb{1}_{\{\tau_j^{\Phi,x} = \infty\}} \le \sum_{j=1}^n \mathbb{1}_{\{\tau_j^{\Phi,y} = \infty\}}.$$

Taking expectations and then the supremum over $\Phi \in \mathcal{A}_n$ proves desired result.

(iii) Let σ be a permutation of $\{1, \ldots, n\}$, and define $Px = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. For every admissible control $\Phi = (\phi^1, \ldots, \phi^n)$, define the permuted control $\widetilde{\Phi} = (\widetilde{\phi}^1, \ldots, \widetilde{\phi}^n)$ by

$$\widetilde{\phi}_t^i := \phi_t^{\sigma(i)}, \qquad t \ge 0.$$

Since the Brownian motions are independent and identically distributed, the law of the controlled process started from Px under $\widetilde{\Phi}$ is the same as that of the permuted process $(X^{\Phi,\sigma(1)}, \ldots, X^{\Phi,\sigma(n)})$ started from x under Φ. Therefore,

$$\mathbb{E}\Big[\prod_{i=1}^n \mathbb{1}_{\{\tau_i^{\widetilde{\Phi}} = \infty\}} \,\Big|\, X_0^{\widetilde{\Phi}} = Px\Big] = \mathbb{E}\Big[\prod_{i=1}^n \mathbb{1}_{\{\tau_i^\Phi = \infty\}} \,\Big|\, X_0^\Phi = x\Big],$$

and similarly for the sum criterion. Taking suprema over admissible controls yields

$$V^n(Px) = V^n(x), \qquad U^n(Px) = U^n(x).$$

This proves the symmetry of V^n and U^n. □

Proof (of Lemma 2). In view of Lemma 1, it suffices to prove the "if" part. Assume first that $b > -1/n$. Consider the constant control $\Phi \in \mathcal{A}_n$ given by

$$\phi_t^i := \frac{1}{n}, \qquad i = 1, \ldots, n, \quad t \ge 0.$$

Then the coordinates are independent and satisfy

$$X_t^{\Phi,i} = x_i + \Big(b + \frac{1}{n}\Big)t + W_t^i, \qquad i = 1, \ldots, n.$$

Hence

$$V^n(x) \geq \prod_{i=1}^n \mathbb{P}\Big[x_i + \Big(b+\frac{1}{n}\Big)t + W_t^i > 0,\ \forall t \geq 0\Big] = \prod_{i=1}^n H\Big(\Big(b+\frac{1}{n}\Big)^+ x_i\Big) > 0.$$

Assume next that $b > -1$. Consider the control $\Phi \in \mathcal{A}_n$ defined by

$$\phi_t^1 := 1, \qquad \phi_t^i := 0, \quad i = 2, \ldots, n, \quad t \geq 0.$$

Then

$$X_t^{\Phi,1} = x_1 + (b+1)t + W_t^1,$$

and therefore

$$U^n(x) \geq \mathbb{P}\big[x_1 + (b+1)t + W_t^1 > 0,\ \forall t \geq 0\big] = H((b+1)^+ x_1) > 0.$$

This proves the lemma. □

Proof (of Lemma 3). The case $b \leq -1/(n-1)$ in (i) is immediate since $V^{n-1} \equiv 0 \equiv V^n$ by Lemma 2. We thus assume $b \geq 0$ in the sequel.

(i) For every admissible control $\Phi \in \mathcal{A}_n$ and every k,

$$\prod_{j=1}^n \mathbb{1}_{\{\tau_j^\Phi = \infty\}} \leq \prod_{j \neq i} \mathbb{1}_{\{\tau_j^\Phi = \infty\}}, \qquad \sum_{j=1}^n \mathbb{1}_{\{\tau_j^\Phi = \infty\}} \leq 1 + \sum_{j \neq i} \mathbb{1}_{\{\tau_j^\Phi = \infty\}}.$$

Taking expectations and then the supremum over $\Phi \in \mathcal{A}_n$, and observing that by restricting to controls with $\phi_t^i \equiv 0$ the remaining coordinates are controlled by an element of $\mathcal{A}_{n-1}$, we obtain for all $z \geq 0$,

$$V^n(x^{-i}, z) \leq V^{n-1}(x^{-i}), \qquad U^n(x^{-i}, z) \leq 1 + U^{n-1}(x^{-i}).$$

Hence, the monotonicity yields

$$V^n(x^{-i}, \infty) \leq V^{n-1}(x^{-i}), \quad U^n(x^{-i}, \infty) \leq 1 + U^{n-1}(x^{-i}). \tag{31}$$

(ii) Fix $\varepsilon > 0$. Choose $\Psi \in \mathcal{A}_{n-1}$ such that

$$\mathbb{E}\Big[\prod_{j \neq i} \mathbb{1}_{\{\tau_j^\Psi = \infty\}} \,\Big|\, X_0^\Psi = x^{-i}\Big] \geq V^{n-1}(x^{-i}) - \varepsilon.$$

For $\eta \in (0,1)$, define $\Phi^\eta \in \mathcal{A}_n$ by

$$\phi_t^{\eta,i} := \eta, \qquad (\phi_t^{\eta,1}, \ldots, \phi_t^{\eta,i-1}, \phi_t^{\eta,i+1}, \ldots, \phi_t^{\eta,n}) := (1-\eta)\Psi_t.$$

Then $\Phi^\eta \in \mathcal{A}_n$ since $\sum_{j=1}^n \phi_t^{\eta,j} \leq \eta + (1-\eta) = 1$. Let X^η be the controlled process under Φ^η started from (x^{-i}, x_i^k). The ith coordinate satisfies

$$\mathrm{d}X_t^{\eta,i} = (b+\eta)\,\mathrm{d}t + \mathrm{d}W_t^i, \qquad X_0^{\eta,i} = x_i^k,$$

and, since $b + \eta > 0$,

$$\mathbb{P}\Big[\tau_i^{\Phi^\eta} = \infty \,\big|\, X_0^{\eta,i} = x_i^k\Big] = 1 - \mathrm{e}^{-2(b+\eta)x_i^k} \xrightarrow[k\to\infty]{} 1. \tag{32}$$

For the remaining coordinates $j \neq i$, the dynamics coincide with those of the $(n-1)$-dimensional system driven by $(1-\eta)\Psi$ started from $\bar{x}$. Denote by $\tau^{(n-1),\eta}$ the minimum ruin time among these $n-1$ coordinates. Since $b \geq 0$ and $\Psi \geq 0$ componentwise, decreasing the control intensity from 1 to $1-\eta$ decreases drifts and therefore decreases survival probabilities; in particular,

$$\mathbb{E}\Big[\prod_{j \neq i} \mathbb{1}_{\{\tau_j^{(1-\eta)\Psi} = \infty\}} \,\Big|\, X_0^{(1-\eta)\Psi} = x^{-i}\Big] \uparrow \mathbb{E}\Big[\prod_{j \neq i} \mathbb{1}_{\{\tau_j^\Psi = \infty\}} \,\Big|\, X_0^\Psi = x^{-i}\Big] \quad \text{as } \eta \downarrow 0.$$

Hence, choosing $\eta > 0$ small enough, we may ensure

$$\mathbb{E}\Big[\prod_{j \neq i} \mathbb{1}_{\{\tau_j^{(1-\eta)\Psi} = \infty\}} \,\Big|\, X_0^{(1-\eta)\Psi} = x^{-i}\Big] \geq V^{n-1}(x^{-i}) - 2\varepsilon. \tag{33}$$

Since the Brownian motions are independent and $\phi_t^{\eta,i} \equiv \eta$ is deterministic, conditioning on the σ-field generated by $(W^j)_{j \neq i}$ yields

$$\mathbb{E}\Big[\prod_{j=1}^{n}\mathbb{1}_{\{\tau_j^{\Phi^\eta}=\infty\}}\,\Big|\,X_0^{\Phi^\eta}=(x^{-i},x_i^k)\Big]=\mathbb{E}\Big[\prod_{j\neq i}\mathbb{1}_{\{\tau_j^{(1-\eta)\Psi}=\infty\}}\,\Big|\,X_0^{(1-\eta)\Psi}=x^{-i}\Big]\,\mathbb{P}\Big[\tau_i^{\Phi^\eta}=\infty\,\big|\,X_0^{\eta,i}=x_i^k\Big].$$

Combining (32)–(33) and letting $k\to\infty$ gives

$$V^n(x^{-i},\infty)=\liminf_{k\to\infty}V^n(x^{-i},x_i^k)\geq V^{n-1}(x^{-i})-2\varepsilon.$$

Since $\varepsilon>0$ is arbitrary,

$$V^n(x^{-i},\infty)\geq V^{n-1}(x^{-i}). \tag{34}$$

Together with (31), this proves (i) for $b\geq 0$.

(iii) Fix $\varepsilon>0$ and choose $\Psi\in\mathcal{A}_{n-1}$ such that

$$\mathbb{E}\Big[\sum_{j\neq i}\mathbb{1}_{\{\tau_j^{\Psi}=\infty\}}\,\Big|\,X_0^{\Psi}=x^{-i}\Big]\geq U^{n-1}(x^{-i})-\varepsilon.$$

Let Φ^η be defined as above. Then

$$\mathbb{E}\Big[\sum_{j=1}^{n}\mathbb{1}_{\{\tau_j^{\Phi^\eta}=\infty\}}\,\Big|\,X_0^{\Phi^\eta}=(x^{-i},x_i^k)\Big]=\mathbb{E}\Big[\mathbb{1}_{\{\tau_i^{\Phi^\eta}=\infty\}}\,\big|\,X_0^{\eta,i}=x_i^k\Big]+\mathbb{E}\Big[\sum_{j\neq i}\mathbb{1}_{\{\tau_j^{(1-\eta)\Psi}=\infty\}}\,\Big|\,X_0^{(1-\eta)\Psi}=x^{-i}\Big].$$

The first term converges to 1 as $k\to\infty$ by (32). For the second term, the same monotonicity-in-η argument yields

$$\mathbb{E}\Big[\sum_{j\neq i}\mathbb{1}_{\{\tau_j^{(1-\eta)\Psi}=\infty\}}\,\Big|\,X_0^{(1-\eta)\Psi}=x^{-i}\Big]\uparrow\mathbb{E}\Big[\sum_{j\neq i}\mathbb{1}_{\{\tau_j^{\Psi}=\infty\}}\,\Big|\,X_0^{\Psi}=x^{-i}\Big]\quad\text{as }\eta\downarrow 0,$$

so choosing $\eta>0$ small enough we can ensure

$$\mathbb{E}\Big[\sum_{j\neq i}\mathbb{1}_{\{\tau_j^{(1-\eta)\Psi}=\infty\}}\,\Big|\,X_0^{(1-\eta)\Psi}=x^{-i}\Big]\geq U^{n-1}(x^{-i})-2\varepsilon.$$

Letting $k\to\infty$ then gives

$$U^n(x^{-i},\infty)=\liminf_{k\to\infty}U^n(x^{-i},x_i^k)\geq 1+U^{n-1}(x^{-i})-2\varepsilon.$$

Since ε is arbitrary, together with (31) we obtain (ii). □

Proof (of Theorem 1). We prove separately the statements for V^n and U^n.

(i) We first show that the push-the-laggard strategy $\overline{\Phi}$ is optimal for V^n when $b\geq 0$. By Proposition 1, the value function V^n belongs to $C^2(E_n)\cap C(\mathbb{R}_+^n)$ and is the unique bounded classical solution of (4), (5), and (7). Let $X=(X^1,\dots,X^n)$ be the controlled process under $\overline{\Phi}$, started from $x\in E_n$, and define

$$\tau:=\min_{1\leq i\leq n}\tau_i,\qquad \sigma_K:=\inf\{t\geq 0:\ \max(X_t)\geq K\},\qquad K>0.$$

Since $\overline{\Phi}$ allocates the whole budget to a coordinate attaining the minimum positive level, Theorem 2 (i) implies that, for every $y\in E_n$,

$$\sum_{j=1}^{n}\overline{\phi}^j(y)\partial_{x_j}V^n(y)=\max(\nabla V^n(y)).$$

Applying Itô's formula to $V^n(X_t)$ on $[0,\tau\wedge\sigma_K]$ yields

$$\begin{aligned}V^n(X_{\tau\wedge\sigma_K})&=V^n(x)+\sum_{j=1}^{n}\int_0^{\tau\wedge\sigma_K}\partial_{x_j}V^n(X_s)\,\mathrm{d}W_s^j\\&\quad+\int_0^{\tau\wedge\sigma_K}\Big(\frac{1}{2}\Delta V^n+b\mathbf{1}_n\cdot\nabla V^n+\sum_{j=1}^{n}\overline{\phi}^j\partial_{x_j}V^n\Big)(X_s)\,\mathrm{d}s.\end{aligned}$$

By the previous identity and the HJB equation (4), the drift term vanishes, so

$$V^n(X_{\tau\wedge\sigma_K})=V^n(x)+\sum_{j=1}^{n}\int_0^{\tau\wedge\sigma_K}\partial_{x_j}V^n(X_s)\,\mathrm{d}W_s^j.$$

Taking expectations gives

$$V^n(x) = \mathbb{E}_x\big[V^n(X_{\tau\wedge\sigma_K})\big]. \tag{35}$$

Since $0 \le V^n \le 1$, we may let $K \to \infty$ and use dominated convergence to obtain

$$V^n(x) = \mathbb{E}_x\big[V^n(X_\tau)\big].$$

Now

$$V^n(X_\tau) = \prod_{i=1}^n \mathbb{1}_{\{\tau_i=\infty\}} \quad \text{a.s.}$$

Indeed, on the event $\{\tau < \infty\}$ one has $X_\tau \in \partial E_n$, hence $V^n(X_\tau) = 0$ by (5); on the event $\{\tau = \infty\}$ all coordinates survive forever, and the recursive boundary condition at infinity implies that $V^n(X_t) \to 1$ as $t \to \infty$. Therefore

$$V^n(x) = \mathbb{E}_x\Big[\prod_{i=1}^n \mathbb{1}_{\{\tau_i=\infty\}}\Big].$$

Since V^n is the supremum over all admissible controls by definition, this proves that $\overline{\Phi}$ is optimal for V^n.

(ii) We next show that $\overline{\Phi}$ is not optimal for U^n. By Theorem 2 (ii), there exists a nonempty open set $O \subset E_n$ such that for every $y \in O$, if $y_i = \min(y)$, then

$$\partial_{x_i} U^n(y) < \max(\nabla U^n(y)).$$

Let $x \in O$. Since O is open, there exists $r > 0$ such that $\overline{B(x,r)} \subset O$. Let $X = (X^1, \dots, X^n)$ be the controlled process under $\overline{\Phi}$, started from x, and let

$$\theta := \inf\{t \ge 0 :\ X_t \notin B(x,r)\}.$$

By Proposition 1, $U^n \in C^2(E_n) \cap C(\mathbb{R}^n_+)$ and solves (4), (6), and (8). Applying Itô's formula to $U^n(X_t)$ on $[0,\theta]$ gives

$$\begin{aligned} U^n(X_\theta) = U^n(x) &+ \sum_{j=1}^n \int_0^\theta \partial_{x_j} U^n(X_s)\,\mathrm{d}W_s^j \\ &+ \int_0^\theta \Big(\frac{1}{2}\Delta U^n + b\mathbf{1}_n\cdot\nabla U^n + \sum_{j=1}^n \overline{\phi}^j \partial_{x_j} U^n\Big)(X_s)\,\mathrm{d}s. \end{aligned}$$

For every $y \in O$, the push-the-laggard strategy allocates the whole budget to a coordinate attaining $\min(y)$, and by the strict inequality above,

$$\sum_{j=1}^n \overline{\phi}^j(y)\partial_{x_j} U^n(y) < \max(\nabla U^n(y)).$$

Since U^n satisfies (4), it follows that

$$\frac{1}{2}\Delta U^n(y) + b\mathbf{1}_n\cdot\nabla U^n(y) + \sum_{j=1}^n \overline{\phi}^j(y)\partial_{x_j} U^n(y) < 0, \qquad y \in O.$$

By continuity of the left-hand side and compactness of $\overline{B(x,r)}$, there exists $\varepsilon > 0$ such that

$$\frac{1}{2}\Delta U^n(y) + b\mathbf{1}_n\cdot\nabla U^n(y) + \sum_{j=1}^n \overline{\phi}^j(y)\partial_{x_j} U^n(y) \le -\varepsilon, \qquad y \in \overline{B(x,r)}.$$

Taking expectations in Itô's formula and using that the stochastic integral is a martingale, we obtain

$$\mathbb{E}_x[U^n(X_\theta)] \le U^n(x) - \varepsilon\,\mathbb{E}_x[\theta].$$

Since $\theta > 0$ a.s., this yields

$$\mathbb{E}_x[U^n(X_\theta)] < U^n(x). \tag{36}$$

Suppose by contradiction that $\overline{\Phi}$ were optimal for U^n at x. By the strong Markov property at time θ, the continuation of $\overline{\Phi}$ after θ would still be optimal from X_θ, and therefore

$$U^n(x) = \mathbb{E}_x\big[U^n(X_\theta)\big].$$

This contradicts (36). Hence $\overline{\Phi}$ is not optimal for U^n. $\square$

Proof (of Proposition 2). The proof is the same verification argument as in the proof of Theorem 1.

Assume first that $b \le -1/n$, and let $v^n \in C^2(E_n) \cap C(\mathbb{R}^n_+)$ be a bounded solution of (4) and (5). Define the feedback control $\Phi \in \mathcal{A}_n$ by

$$\phi^i_t := \mathbb{1}_{\{\partial_{x_i} v^n(X_t) = \max(\nabla v^n(X_t))\}},$$

with the usual tie-breaking rule. Exactly as in the proof of Theorem 1, Itô's formula gives

$$v^n(x) = \mathbb{E}_x\big[v^n(X_{\tau\wedge\sigma_K})\big].$$

Now, setting $Y_t := \sum_{j=1}^n X^j_t$, one has

$$\mathrm{d}Y_t = \Big(nb + \sum_{j=1}^n \phi^j_t\Big)\mathrm{d}t + \sum_{j=1}^n \mathrm{d}W^j_t,$$

and since $nb + 1 \le 0$, it follows that $\tau < \infty$ a.s. Letting $K \to \infty$ and using boundedness of v^n, we obtain

$$v^n(x) = \mathbb{E}_x[v^n(X_\tau)].$$

Because $X_\tau \in \partial E_n$ and $v^n = 0$ on ∂E_n, we conclude that $v^n(x) = 0$ for all $x \in E_n$. Hence $v^n \equiv 0$ on $\mathbb{R}^n_+$ by continuity.

The proof for u^n is identical. If $b \le -1$ and $u^n \in C^2(E_n) \cap C(\mathbb{R}^n_+)$ is a bounded solution of (4) and (6), define the same maximizing feedback control. Then Itô's formula gives

$$u^n(x) = \mathbb{E}_x\big[u^n(X_{\tau\wedge\sigma_K})\big].$$

Since now each coordinate has drift bounded above by 0, one has $\tau < \infty$ a.s. Letting $K \to \infty$ yields

$$u^n(x) = \mathbb{E}_x[u^n(X_\tau)].$$

Using the boundary condition (6) and arguing by induction on n, exactly as in the original proof, gives $u^n(X_\tau) = 0$, hence $u^n \equiv 0$. □

Proof (of Lemma 7). We begin by recalling the local function spaces used below.

For an open set $O \subset \mathbb{R}^n$ and $p \in (1,\infty)$, the Sobolev space $W^{2,p}(O)$ consists of all functions $u \in L^p(O)$ whose weak derivatives up to order two belong to $L^p(O)$, endowed with the norm

$$\|u\|_{W^{2,p}(O)} := \sum_{|\beta|\le 2} \|D^\beta u\|_{L^p(O)}.$$

We write

$$W^{2,p}_{\mathrm{loc}}(E_n) := \big\{u : E_n \to \mathbb{R} :\ u \in W^{2,p}(K) \text{ for every compact } K \Subset E_n\big\}.$$

Similarly, for $\alpha \in (0,1)$ and an open set $O \subset \mathbb{R}^n$, the Hölder space $C^{2,\alpha}(O)$ consists of all functions $u \in C^2(O)$ such that

$$\|u\|_{C^{2,\alpha}(O)} := \sum_{|\beta|\le 2} \|D^\beta u\|_{L^\infty(O)} + \sum_{|\beta|=2} [D^\beta u]_{C^{0,\alpha}(O)} < \infty,$$

where

$$[w]_{C^{0,\alpha}(O)} := \sup_{\substack{x,y\in O\\ x\ne y}} \frac{|w(x)-w(y)|}{|x-y|^\alpha}.$$

We write

$$C^{2,\alpha}_{\mathrm{loc}}(E_n) := \big\{u : E_n \to \mathbb{R} :\ u \in C^{2,\alpha}(K) \text{ for every compact } K \Subset E_n\big\}.$$

Fix compact sets

$$K \Subset K' \Subset E_n.$$

By Lemma 6, V is Lipschitz on K'. Hence, by Rademacher's theorem, V is differentiable almost everywhere on K' and

$$\nabla V \in L^\infty(K').$$

Recall that

$$G(q) := -b\,\mathbf{1}_n \cdot q - \max(q), \qquad q \in \mathbb{R}^n.$$

Since G is globally Lipschitz, the function

$$f := G(\nabla V)$$

is well defined almost everywhere on K' and satisfies

$$f \in L^\infty(K').$$

Because V is a viscosity solution of (14), the equation may be rewritten as

$$\Delta V = 2f$$

in the viscosity sense on K'. Since the Laplacian is uniformly elliptic and $f \in L^\infty(K')$, the equivalence between viscosity and distributional solutions for linear uniformly elliptic equations with bounded right-hand side implies that V is a distributional solution of

$$\Delta V = 2f \qquad \text{in } K'.$$

A convenient reference is Caffarelli–Cabré [13, Prop. 2.9].

Let $p \in (1, \infty)$ be arbitrary. By the interior Calderón–Zygmund estimate for Poisson's equation, see Gilbarg–Trudinger [16, Thm. 9.11], there exists a constant $C_{K,K',p} > 0$ such that

$$\|V\|_{W^{2,p}(K)} \leq C_{K,K',p}\Big(\|V\|_{L^p(K')} + \|f\|_{L^p(K')}\Big).$$

Since V is bounded and $f \in L^\infty(K')$, the right-hand side is finite. Therefore

$$V \in W^{2,p}(K).$$

As $K \Subset E_n$ is arbitrary, we conclude that

$$V \in W^{2,p}_{\text{loc}}(E_n) \qquad \text{for every } p \in (1, \infty).$$

Choose now $p > n$ and set

$$\alpha := 1 - \frac{n}{p} \in (0, 1).$$

By the Sobolev embedding theorem, see Gilbarg–Trudinger [16, Thm. 7.26],

$$W^{2,p}(K) \hookrightarrow C^{1,\alpha}(K).$$

Hence

$$V \in C^{1,\alpha}(K).$$

Since G is globally Lipschitz, it follows that

$$f = G(\nabla V) \in C^{0,\alpha}(K).$$

Finally, applying the interior Schauder estimate for Poisson's equation, see Gilbarg–Trudinger [16, Thm. 6.2 and Thm. 6.6], we obtain

$$V \in C^{2,\alpha}(K).$$

As $K \Subset E_n$ is arbitrary, this shows that

$$V \in C^{2,\alpha}_{\text{loc}}(E_n).$$

In particular, $V \in C^2(E_n)$. Since Proposition 3 already gives

$$V \in C(\mathbb{R}^n_+),$$

we conclude that

$$V \in C^2(E_n) \cap C(\mathbb{R}^n_+).$$

□

References

1. K. B. Ahsan, M. Alam, D. G. Morel, and M. Karim. Emergency department resource optimisation for improved performance: a review. *J. Ind. Eng. Int.*, 15(Suppl 1):253–266, 2019.
2. D. Aldous. "Up the River" game story. 2002. Available at http://www.stat.berkeley.edu/ aldous/Research/OP/river.pdf.
3. B. Ata, J. M. Harrison, and N. Si. Drift control of high-dimensional reflected Brownian motion: a computational method based on neural networks. *Stoch. Syst.*, 15(2):111–146, 2025.
4. R. Atar and T. Ichiba. Rank-based stochastic differential inclusions and diffusion limits for a load balancing model. 2024. arXiv:2409.15121.
5. G. Baker, B. Hambly, and P. Jettkant. Particle systems and mckean–vlasov dynamics with singular interaction through local times. 2025. arXiv:2503.08837.
6. S. Banerjee and A. Budhiraja. Domains of attraction of invariant distributions of the infinite Atlas model. *Ann. Probab.*, 50(4):1610–1646, 2022.
7. S. Banerjee and A. Budhiraja. Extremal invariant distributions of infinite Brownian particle systems with rank dependent drifts. *Probab. Theory Related Fields*, 190(1-2):79–117, 2024.
8. S. Banerjee, A. Budhiraja, and B. Estevez. Load balancing in parallel queues and rank-based diffusions. 2023. arXiv:2302.10317.
9. A. D. Banner, R. Fernholz, and I. Karatzas. Atlas models of equity markets. *Ann. Appl. Probab.*, 15(4):2296–2330, 2005.
10. E. Bayraktar, G. Guo, W. Tang, and Y. P. Zhang. Systemic robustness: a mean-field particle system approach. *Math. Finance*, 35(4):727–744, 2025.
11. B. Bouchard and N. Touzi. Weak dynamic programming principle for viscosity solutions. *SIAM J. Control Optim.*, 49(3):948–962, 2011.
12. M. Cabezas, A. Dembo, A. Sarantsev, and V. Sidoravicius. Brownian particles with rank-dependent drifts: out-of-equilibrium behavior. *Comm. Pure Appl. Math.*, 72(7):1424–1458, 2019.
13. L. A. Caffarelli and X. Cabré. *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.
14. M. Dauge. *Elliptic boundary value problems on corner domains: Smoothness and asymptotics of solutions*, volume 1341 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.
15. D. Duma and R. Aringhieri. Real-time resource allocation in the emergency department: A case study. *Omega*, 117:102844, 2023.
16. D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
17. X. Guo, W. Tang, and R. Xu. A class of stochastic games and moving free boundary problems. *SIAM J. Control Optim.*, 60(2):758–785, 2022.
18. T. Ichiba, I. Karatzas, and M. Shkolnikov. Strong solutions of stochastic equations with rank-based coefficients. *Probab. Theory Related Fields*, 156(1-2):229–248, 2013.
19. H. Ishii and P.-L. Lions. Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. *J. Differential Equations*, 83(1):26–78, 1990.
20. B. Jourdain and J. Reygner. Propogation of chaos for rank-based interacting diffusions and long time behaviour of a scalar quasilinear parabolic equation. *Stoch. Partial Differ. Equ. Anal. Comput.*, 1(3):455–506, 2013.
21. H. P. McKean and L. A. Shepp. The advantage of capitalism vs. socialism depends on the criterion. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 328:160–168, 279–280, 2005.
22. S. Pal and J. Pitman. One-dimensional Brownian particle systems with rank-dependent drifts. *Ann. Appl. Probab.*, 18(6):2179–2207, 2008.
23. M. d. C. Patrão Neves. Criteria for ethical allocation of scarce healthcare resources: Rationing vs. rationalizing in the treatment for the elderly. *Philosophies*, 7(6):123, 2022.
24. G. Persad, A. Wertheimer, and E. J. Emanuel. Principles for allocation of scarce medical interventions. *Lancet*, 373(9661):423–431, 2009.
25. A. Sarantsev and L.-C. Tsai. Stationary gap distributions for infinite systems of competing brownian particles. *Electron. J. Probab*, 22(56):1–20, 2017.
26. W. Tang and L.-C. Tsai. Optimal surviving strategy for drifted Brownian motions with absorption. *Ann. Probab.*, 46(3):1597–1650, 2018.
27. A. Zeneli, G. Brandi, G. Di Pasquale, D. Orlandini, P. De Carolis, F. Bravi, F. Pugliese, E. Poluzzi, F. Catena, and F. Giovanardi. Identifying ethical values for guiding triage decisions during the covid-19 pandemic: an italian ethical committee perspective using delphi methodology. *BMJ Open*, 11(5):e043239, 2021.