IEOR 4106: Introduction to Operations Research: Stochastic Models

Spring 2011, Professor Whitt,

Renewal Theory, Chapter 7, Thursday, April 7

The Inspection Paradox, the Excess and the Age

Taking the M60 bus to Laguardia Airport. Suppose that you go out to the bus stop on Broadway at the campus gate on 116^{th} Street in order to take the bus to Laguardia Airport. Suppose that you go at a random time, without knowing anything about the specific bus schedule. Suppose that you know that, for the general time of day you are going, that the average time between successive M60 buses is 20 minutes. Thus you deduce that your waiting time, on average, should be $20/2 = 10$ minutes. However, on the five times you went, your average wait was 18 minutes. Is this yet another example illustrating that you have incredibly bad luck, much worse than everyone else?

A survey of tourists in Morocco. Homework Exercise 7.51

These examples illustrate **the inspection paradox**, which is presented in Section 7.7 of the book. Homework exercise 51 is related to this. (That is the study of tourism in Morocco.) We talked about how long a person would have to wait for the next bus, if he were to go out to the bus stop at a random time, unaware of the bus schedule. We emphasized that the distribution of the time until the next bus, from the perspective of an arrival at an arbitrary time, is in general different than the distribution of the time between successive bus arrivals. This is true in great generality. For this chapter, though, we assume that the successive bus arrivals form a renewal process, which means that the successive intervals between successive arrivals form a sequence of IID random variables, each distributed as the random variable X with cdf F.

1. We also discussed the **age of a renewal process** and the **excess (lifetime) of a renewal process**, depicted in Figure 1; see Examples 7.16 and 7.17 on pages 443-447 of the tenth edition (or pages 440-441 (of the ninth edition) and Examples 7.23 and 7.24 on pages 452-454 (or pages 447-448 of the ninth edition).

Let F be the cdf of the time between successive arrivals. Suppose that the cdf F has mean m. Then the cdf of the time until the next bus, from the perspective of an arrival at an arbitrary time, is

$$F_e(x) = \frac{1}{m} \int_0^x (1 - F(u))\, du \ ;$$

see Examples 7.23 and 7.24 in the book. The cdf F_e is the *equilibrium-excess cdf* or the stationary-excess cdf or the equilibrium-residual-lifetime cdf. That distribution is the same as the distribution of the age, discussed in Example 7.23 on p. 430.

If we understand $F_e(x)$ as the long-run proportion of time that the age (or excess life) is less than or equal to x, then we can derive the cdf F_e using **renewal reward theory**, just like we found the earning rate of the small-town traffic cop. That is done in Examples 7.23 and 7.24 (and in class).

You want to use the tail-integral expression for the mean:

$$E[X] = \int_0^\infty P(X > y)\, dy$$

see bottom of p. 604 in the ninth edition (p. 580 of the eighth edition).

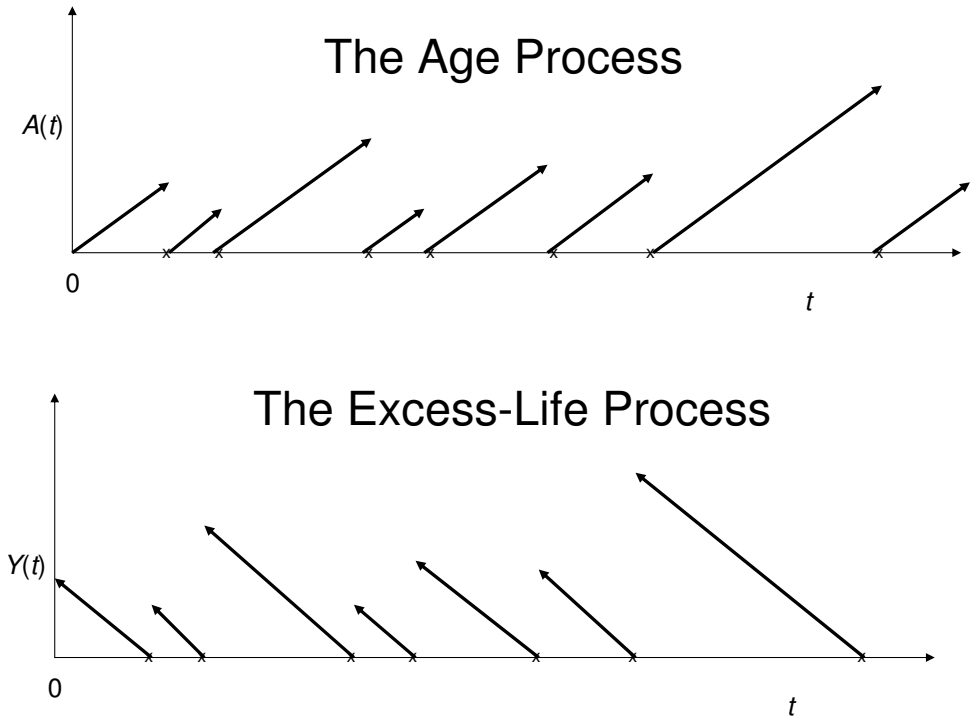


Figure 1: The age and residual-lifetime processes associated with a renewal process.

We might just be interested in the mean of the cdf F_e. In other words, we might be interested in the long-run average. The average age and average excess (or residual life or remaining life) are discussed in Example 7.15. and Example 7.16. Note that the average age and average excess depends on the second moment of the time between successive points, as well as the mean, as given in formula (7.14). I suggested rewriting (7.14) as

$$E[X_e] \equiv \text{average value of excess} = \text{average age} = \frac{E[X^2]}{2E[X]} = \frac{(c^2+1)}{2} \times E[X] \ ,$$

where

$$c^2 \equiv \frac{Var(X)}{E[X]^2}$$

is the *squared coefficient of variation* (SCV). The SCV measures variability independent of the mean or, equivalently, independent of scale. Note that

$$c^2_{bX} = c^2_X$$

for any positive constant b and any nonnegative random variable X. For an exponential random variable X, $c^2_X = 1$; for a deterministic random variable (P(X=d) = 1), $c^2_X = 0$.

If we understand $E[X_e]$ as the long-run average age (or excess life), then we can derive the long-run average $E[X_e]$ using **renewal reward theory**, just like we found the earning rate of the small-town traffic cop. That is done in Examples 7.16 and 7.17.