IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2006, Professor Whitt,

Renewal Theory, Chapter 7, Tuesday, November 21

Age, Excess and the Inspection Paradox

We first discussed the age and excess of a renewal process; see Examples 7.15 and 7.16 on pages 423-425 and Examples 7.22 and 7.23 on pages 430-432. Then we discussed the inspection paradox, which is presented in Section 7.7 of the book. Homework exercise 51 is related to this. We talked about how long a person would have to wait for the next bus, if he were to go out to the bus stop at a random time, unaware of the bus schedule. We emphasized that the distribution of the time until the next bus, from the perspective of an arrival at an arbitrary time, is in general different than the distribution of the time between successive bus arrivals. This is true in great generality. For this chapter, though, we assume that the successive bus arrivals form a renewal process, which means that the successive intervals between successive arrivals form a sequence of IID random variables, each distributed as the random variable X with cdf F.

Let F be the cdf of the time between successive arrivals. Suppose that the cdf F has mean m. Then the cdf of the time until the next bus, from the perspective of an arrival at an arbitrary time, is

$$F_e(x) = \frac{1}{m}\int_0^x (1 - F(u))\, du \ ;$$

see Examples 7.22 and 7.23 in the book. The cdf F_e is the *equilibrium-excess cdf* or the stationary-excess cdf or the equilibrium-residual-lifetime cdf. That distribution is the same as the distribution of the age, discussed in Example 7.22 on p. 430.

We might just be interested in the mean of the cdf F_e. In other words, we might be interested in the long-run average. The average age and average excess (or residual life or remaining life) are discussed in Example 7.15. and Example 7.16. Note that the average age and average excess depends on the second moment of the time between successive points, as well as the mean, as given in formula (7.14). I suggested rewriting (7.14) as

$$\text{average value of excess} = \text{average age} = \frac{E[X^2]}{2E[X]} = \frac{(c^2+1)}{2} \times E[X] \ ,$$

where

$$c^2 \equiv \frac{Var(X)}{E[X]^2}$$

is the *squared coefficient of variation* (SCV). The SCV measures variability independent of the mean or, equivalently, independent of scale. Note that

$$c^2_{bX} = c^2_X$$

for any positive constant b and any nonnegative random variable X. For an exponential random variable X, $c^2_X = 1$; for a deterministic random variable (P(X=d) = 1), $c^2_X = 0$.