**IEOR 3106: Introduction to Operations Research: Stochastic Models**

**Fall 2006, Professor Whitt**

**Solutions to Homework Assignment 1b, due Thursday, September 14**

**Probability Review**

Read Chapter 1 and Sections 2.1-2.4 of Chapter 2 in the textbook, *Introduction to Probability Models*, eighth edition, by Sheldon Ross.

Problems from Chapters 1 and 2 of Ross. In all homework, show your work.

Problem 1.13 (Hint: This problem draws on Problems 1.11 and 1.12.)

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1.12 Let $E$ and $F$ be mutually exclusive events in a sample space. That means that $EF \equiv E \cap F = \phi$, i.e., the two sets have no common elements. Suppose that the experiment is repeated until one of the events $E$ or $F$ occurs. Let $p$ be the probability that $E$ occurs before $F$. Intuitively,

$$p = \frac{P(E)}{P(E) + P(F)} ,$$

but how do we prove it? One way is to condition on the outcome of the first experiment. The outcome can be one of three things: it can be $E$; it can be $F$; or it can be neither of these two. In the event that it is neither $E$ nor $F$, we start over. That leads to the following equation:

$$p = 1 \times P(E) + 0 \times P(F) + p \times (1 - P(E) - P(F)) .$$

Solving this linear equation gives the desired formula above.

1.13 This draws on Problems 1.11 and 1.12. Again, like the solution to 1.12 above, we can calculate the desired probability by conditioning on the outcome of the first roll, and multiply by the probabilities. Doing that, we get

$$P(win) = \sum_{i=2}^{i=12} P(win|i)P(i)$$

where $P(win|i) = 0$ for $i = 2, \quad 3 \quad \text{and} \quad 12$, $P(win|i) = 1$ for $i = 7 \quad \text{and} \quad 11$ and, by Problem 1.12,

$$P(win|i) = \frac{i-1}{5+i} \quad \text{for} \quad 3 \leq i \leq 6$$

and

$$P(win|i) = \frac{13-i}{19-i} \quad \text{for} \quad 8 \leq i \leq 10$$

That yields a probability of $0.4929\ldots$ of winning.

If, instead an initial roll of 3 is not a loss, but leads to a repeated roll, treated like an initial 4, then there is an extra 0.013888 chance of winning making the probability 0.50681. That feature makes the difference between having the probability less than 1/2 or greater than 1/2.

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Problem 1.20

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For three dice, there are $6 \times 6 \times 6 = 216$ possible outcomes, ranging from $(1, 1, 1)$ to $(6, 6, 6)$. The key is to develop an efficient way of counting. Let $E$ be the event that exactly two of the three numbers are the same; let $S$ be the event that all three numbers are the same, and let $D$ be the event that all three numbers are different. Observe that these three events partition the sample space, i.e., they are mutually exclusive events whose union is the full space. Thus

$$P(E) + P(S) + P(D) = 1 .$$

It is easy to see that $P(S) = 6/216 = 1/36$; for $D$ we have 6 possibilities for the first die, 5 possibilities for the second die, and 4 possibilities for the third die, so that

$$P(D) = \frac{6 \times 5 \times 4}{216} = \frac{120}{216} = \frac{20}{36} .$$

Hence,

$$P(E) = 1 - P(S) - P(D) = 1 - \frac{1}{36} - \frac{20}{36} = \frac{15}{36} = \frac{5}{12} .$$

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Problem 1.22

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The calculation becomes easy if we view it in a good way. One might observe that the experiment has the form of tennis once deuce has been reached or table tennis (ping pong) after the score of $20 - 20$ has been reached, so this is a situation that you may well have encountered previously.

Here is a good way to view it: Let successive trials consist of successive pairs of points. That is, we consider what happens after 2 plays, after 4 plays, and so on. We do that, because we observe that the game can only end after an even number of plays. Moreover, with this definition of a "trial," the sequence of games becomes a sequence of Bernoulli trials (a sequence of independent and identically distributed (IID) Bernoulli random variables; see Section 2.2.1. The probability that each player wins one point on any trial is $q = 2p(1 - p)$. A total of $2n$ points are played if the first $(n - 1)$ trials all results in each player winning one point, and the $n^{\text{th}}$ trial results in one of the players winning two points. We apply the independence to calculate the desired results.

The probability that $2n$ points are needed is thus $(2p(1 - p))^{n-1}(p^2 + (1 - p)^2)$, $n \geq 1$.

The probability that $A$ eventually wins is

$$\begin{aligned} P(A \quad \text{wins}) &= (\sum_{n=1}^{\infty} (2p(1-p))^{n-1}) \times p^2 \\ &= \frac{p^2}{1 - 2p(1-p)} , \end{aligned}$$

using the well known property of geometric series:

$$\sum_{i=0}^{\infty} x^n = \frac{1}{1 - x}$$

for $0 < x < 1$.

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Problem 1.43

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Here is another application of Bayes' Theorem, again like the examples in Section 1.6. Let $i$ be the event that coin $i$ is selected. As usual, let $H$ and $T$ denote the events head and tail. We are told that $P(H|i) = i/10$; we want to calculate the conditional probability the other way. We get

$$\begin{aligned} P(i=5|H) &= \frac{P(H \cap 5)}{P(5)} \\ &= \frac{P(H|5)P(5)}{\sum_{i=1}^{10} P(H|i)P(i)} \\ &= \frac{5/100}{\sum_{i=1}^{10} i/100} \\ &= \frac{5}{\sum_{i=1}^{10} i} \\ &= \frac{5}{55} = \frac{1}{11} = 0.090909\ldots \end{aligned}$$

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Problem 2.20

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This is a multinomial distribution. Let $T$ be the number of ordinary television sets purchsed; let $C$ be the number of color television sets purchased; and let $B$ be the number of customers browsing. Then

$$P(T=2, C=2, B=1|T+C+B=5) = \frac{5!}{2!2!1!}(0.2)^2(0.3)^2(0.5)^1 = 0.054$$

This could be approached by using the binomial distribution twice. First we consider the purchasers of ordinary TV's and all others. Then we divide up the others into purchasers of color TV's and browsers. That leads to the product of two binomial distributions, as follows:

$$P(T=2, C=2, B=1|T+C+B=5) = P(T=2, T^c=3|T+C+B=5)P(C=2, B=1|C+B=3)\ ,$$

yielding the same result after calculation.

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Problem 2.32

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This problem is about approximating the binomial distribution by the Poisson distribution. Directly, the problem can be solved by the binomial distribution, but the binomial distribution with large $n$ and small $p$, with $np = \lambda$, is approximately Poisson with mean $\lambda$. Let $N$ be the number of prizes won. Then, first exactly and then approximately,

$$\begin{aligned} P(N=k) &= b(k;n,p) = b(k,50,0.01) = \frac{50!}{k!(50-k)!}(0.01)^k(0.99)^{50-k} \\ &\approx Poisson(k; \lambda = np) = Poisson(k; 0.5) = \frac{e^{-0.5}(0.5)^k}{k!} \end{aligned}$$

so that

$$P(N \geq 1) = 1 - P(N = 0) \approx 1 - e^{0.5} = 0.394$$
$$P(N = 1) = P(N = 1) \approx e^{0.5}(0.5)^1 = 0.303$$
$$P(N \geq 2) = 1 - P(N = 0) - P(N = 1) \approx 0.394 - 0.303 = 0.091$$

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Problem 2.34

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To be a probability density function, its integral must be 1. So we must have

$$c \int_0^2 (4x - 2x^2)\, dx = c(2x^2 - 2x^3/3) = 1 \ ,$$

which implies that $c = 3/8$. Then

$$P(1/2 < X < 3/2) = \frac{3}{8} \int_{1/2}^{3/2} (4x - 2x^2)\, dx = \frac{11}{16} \ .$$

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Problem 2.43

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This problem gives practice in the use of indicator random variables, the $X_i$ here.

(a)

$$X = \sum_{i=1}^{n} X_i$$

(b)

$$EX_i = P(X_i = 1) = P(\text{red ball chosen before all black balls}) = \frac{1}{m+1} \ ,$$

so

$$EX = \frac{n}{m+1} \ .$$

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Problem 2.50

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Use definition of the variance.

(i)

$$Var(cX) = E[(cX - E[cX])^2] = E[c^2(X - EX)^2] = c^2 E[(X - EX)^2] = c^2 Var(X)$$

(ii)

$$Var(c + X) = E[(c + X - E[c + X])^2] = E[(X - EX)^2] = Var(X) \ .$$